SAT and SMT Solving

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Outline

- Summary of Last Week
- Satisfiability Modulo Theories
- DPLL(T)
- Using SMT Solvers with Theories

Definitions

for unsatisfiable CNF formula φ given as set of clauses

- $\psi \subseteq \varphi$ such that $\bigwedge_{C \in \psi} C$ is unsatisfiable is unsatisfiable core (UC) of φ
- minimal unsatisfiable core ψ is UC such that every subset of ψ is satisfiable
- SUC (minimum unsatisfiable core) is UC such that $|\psi|$ is minimal

Remark

SUC is always minimal unsatisfiable core

Definition (Resolution Graph)

directed acyclic graph $G = (V, E)$ is resolution graph for set of clauses φ if

1. $V = V_i \uplus V_c$ is set of clauses and $V_i = \varphi$,
2. V_i nodes have no incoming edges,
3. there is exactly one node $\square$ without outgoing edges,
4. $\forall C \in V_c \; \exists$ edges $D \to C$, $D' \to C$ such that C is resolvent of D and D', and
5. there are no other edges.

Algorithm minUnsatCore(φ)

Input: unsatisfiable formula φ
Output: minimal unsatisfiable core of φ

build resolution graph $G = (V_i \uplus V_c, E)$ for φ
while $\exists$ unmarked clause in V_i **do**
    $C \leftarrow$ unmarked clause in V_i
    if SAT($\overline{Reach_G(C)}$) **then** ▷ subgraph without C satisfiable?
        mark C ▷ C is UC member
    else
        build resolution graph $G' = (V_i' \uplus V_c', E')$ for $\overline{Reach_G(C)}$
        $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
        $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
        $G \leftarrow (V_i \cup V_c, E)$
        $G \leftarrow G|_{BReach_G(\Box)}$ ▷ restrict to nodes with path to $\Box$
return V_i

Theorem

if φ unsatisfiable then minUnsatCore(φ) is minimal unsatisfiable core of φ

Definition (Partial minUNSAT)

pminUNSAT(χ, φ) is minimal $|\psi|$ such that $\psi \subseteq \varphi$ and $\chi \wedge \bigwedge_{C \in \psi} \neg C$ satisfiable

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Algorithm FuMalik(χ, φ)

Input: clause set φ and satisfiable clause set χ

$cost \leftarrow 0$
while $\neg$SAT$(\chi \cup \varphi)$ **do**
  $UC \leftarrow$ unsatCore$(\chi \cup \varphi)$ ▷ must be minimal
  $B \leftarrow \varnothing$
  for $C \in UC \cap \varphi$ **do** ▷ loop over soft clauses in core
    $b \leftarrow$ new blocking variable
    $\varphi \leftarrow \varphi \setminus \{C\} \cup \{C \vee b\}$
    $B \leftarrow B \cup \{b\}$
  $\chi \leftarrow \chi \cup \mathtt{CNF}(\sum_{b \in B} b = 1)$ ▷ cardinality constraint is hard
  $cost \leftarrow cost + 1$
return $cost$

Theorem

FuMalik$(\chi, \varphi) =$ pminUNSAT(χ, φ)

$|\varphi| =$ pminUNSAT$(\chi, \varphi) +$ pmaxSAT(χ, φ)

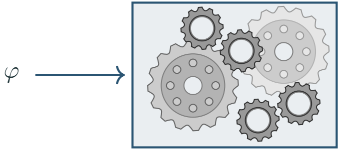
Outline

- Summary of Last Week
- Satisfiability Modulo Theories
  - Recap: First-Order Logic
  - Eager and Lazy Paradigms
- DPLL(T)
- Using SMT Solvers with Theories

SMT Solving

input: formula φ involving theory T

output:

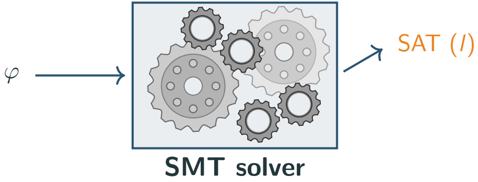


SMT solver

SMT Solving

input: formula φ involving theory T

output: SAT + valuation v such that $v(\varphi) = T$ if φ is T-satisfiable



SMT Solving

input:	formula φ involving theory T	
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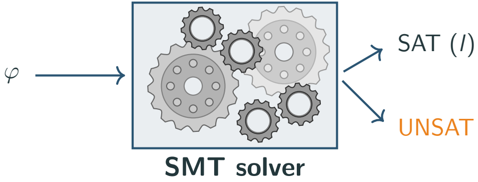
	UNSAT	otherwise



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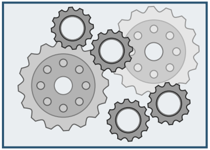
φ $\longrightarrow$

$a + b \geqslant c \vee (a = 0 \wedge p)$



SMT solver

SAT (I) $\quad v(a) = 3 \quad v(b) = 0 \quad v(c) = 0 \quad v(p) = \mathrm{T}$

UNSAT

Example (Common theories)

- arithmetic $\qquad 2a + b \geqslant c \vee (a - b = c + 3 \wedge p)$

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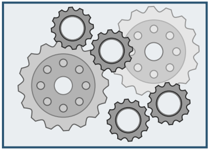

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UNSAT

Example (Common theories)

- arithmetic $2a + b \geqslant c \vee (a - b = c + 3 \wedge p)$
- uninterpreted functions $\mathrm{f}(x, y) \neq \mathrm{f}(y, x) \wedge \mathrm{g}(\mathrm{a}) = \mathrm{a} \to \mathrm{g}(\mathrm{f}(x, x)) = \mathrm{g}(y)$

SMT Solving

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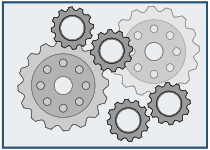

SMT solver

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→ UNSAT

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- uninterpreted functions $\qquad \mathrm{f}(x, y) \neq \mathrm{f}(y, x) \wedge \mathrm{g}(\mathrm{a}) = \mathrm{a} \rightarrow \mathrm{g}(\mathrm{f}(x, x)) = \mathrm{g}(y)$
- bit vectors $\qquad ((\mathrm{zext}_{32}\ a_8) + b_{32}) \times c_{32} >_u 0_{32}$

Definitions (Signature)

- signature $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ consists of

First-Order Logic: Syntax

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- signature $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ consists of
  - set of function symbols $\mathcal{F}$
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  where each symbol is associated with fixed arity (i.e., number of arguments)

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- function/predicate symbols with arity
  - 1 are called unary
  - 2 are called binary
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Definitions (Formulas)

- Σ-terms t are built according to grammar

  $$t \quad ::= \quad x \mid c \mid f(\underbrace{t, \dots, t}_{n})$$

  for constant $c \in \mathcal{F}$, function symbol $f \in \mathcal{F}$ of arity $n > 0$, and variable $x \in X$

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infinite set of variables X

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- Σ-formulas are built according to grammar

  $$\varphi \quad ::= \quad Q \mid P(\underbrace{t, \dots, t}_{n}) \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x.\, \varphi \mid \exists x.\, \varphi$$

  for constant $Q \in \mathcal{P}$, predicate symbol $P \in \mathcal{P}$ of arity $n > 0$, and Σ-terms t

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  for constant $Q \in \mathcal{P}$, predicate symbol $P \in \mathcal{P}$ of arity $n > 0$, and Σ-terms t
- variable x is free in φ if it is not bound by quantifier above

Notation
write f/n or P/n to express that f or P have arity n

Example

- let $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{F} := \{\mathsf{a}/0,\ \mathsf{b}/0,\ \mathsf{f}/1,\ \mathsf{g}/2\}$

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- the following are Σ-terms:
  - a, b, and $\mathsf{f}(\mathsf{a})$
  - x, y, and z — x, y, z are variables
  - $\mathsf{g}(\mathsf{a}, \mathsf{f}(x))$ and $\mathsf{g}(\mathsf{g}(\mathsf{a}, y), \mathsf{f}(\mathsf{b}))$

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  - $\neg(\mathsf{P}(\mathsf{a}) \wedge \mathsf{P}(x) \wedge \mathsf{P}(y))$

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  - $\exists x.\, \mathsf{P}(x)$

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  - $\mathsf{P}(x) \vee (\exists x.\, \mathsf{P}(x) \wedge \mathsf{f}(y) = x)$

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  - $\forall x\, y\, z.\ (x = y \wedge y = z \ \rightarrow\ x = z)$
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- the following are Σ-formulas (free variables highlighted):
  - $\mathsf{a} = \mathsf{f}(\mathsf{b})$
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  - $\neg(\mathsf{P}(\mathsf{a}) \land \mathsf{P}(x) \land \mathsf{P}(y))$
  - $\exists x.\, \mathsf{P}(x)$
  - $\mathsf{P}(x) \lor (\exists x.\, \mathsf{P}(x) \land \mathsf{f}(y) = x)$
  - $\forall x\, y\, z.\, (x = y \land y = z \ \to\ x = z)$
  - $\forall x\, y.\, (x = y \ \to\ y = x)$

write $\varphi \to \psi$ for $\neg\varphi \lor \psi$

First-Order Logic: Semantics

Definition (Model)

model $\mathcal{M}$ for signature $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ consists of

1. non-empty set A (universe of concrete values)
2. function $f^{\mathcal{M}} \colon A^n \to A$ for every n-ary $f \in \mathcal{F}$
3. set of n-tuples $P^{\mathcal{M}} \subseteq A^n$ for every n-ary $P \in \mathcal{P}$

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Example

function and predicate symbols $\mathcal{F} = \{\mathsf{f}/1, \mathsf{a}/0\}$ and $\mathcal{P} = \{\mathsf{R}/2\}$

1. model $\mathcal{M}_1$: universe $A_1 = \mathbb{N}$
   $\mathsf{f}^{\mathcal{M}_1}(x) = 2x + 1$
   $\mathsf{a}^{\mathcal{M}_1} = 0$
   $\mathsf{R}^{\mathcal{M}_1} = \{(x, y) \mid x < y\}$

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   $\mathsf{R}^{\mathcal{M}_1} = \{(x, y) \mid x < y\}$
2. model $\mathcal{M}_2$: universe A_2 is set of all Twitter users
   $\mathsf{f}^{\mathcal{M}_2}(x) =$ last person who started following x (or x if no follower)
   $\mathsf{a}^{\mathcal{M}_2} =$ `@elonmusk`
   $\mathsf{R}^{\mathcal{M}_2} = \{(x, y) \mid x \text{ follows } y\}$

Definitions

- environment for model $\mathcal{M} = \langle A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}} \rangle$ is mapping $I \colon X \to A$

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- environment for model $\mathcal{M} = \langle A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}} \rangle$ is mapping $I \colon X \to A$
- value $t^{\mathcal{M},I}$ of term t in model $\mathcal{M}$ wrt environment I is defined inductively:
$$t^{\mathcal{M},I} = \begin{cases} I(t) & \text{if } t \text{ is a variable} \\ f^{\mathcal{M}}(t_n^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

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- for environment I, variable x and $a \in A$, extended environment $I[x \mapsto a]$ is

$$(I[x \mapsto a])(y) = \begin{cases} a & \text{if } x = y \\ I(y) & \text{otherwise} \end{cases}$$

Definitions

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$$(I[x \mapsto a])(y) = \begin{cases} a & \text{if } x = y \\ I(y) & \text{otherwise} \end{cases}$$

- satisfaction relation $\mathcal{M} \models_I \varphi$ is defined inductively:

$$\mathcal{M} \models_I \varphi \quad \iff \quad \begin{cases} (t_n^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I}) \in P^{\mathcal{M}} & \text{if } \varphi = P(t_1, \dots, t_n) \\ \mathcal{M} \not\models_I \psi & \text{if } \varphi = \neg\psi \\ \mathcal{M} \models_I \varphi_1 \text{ and } \mathcal{M} \models_I \varphi_2 & \text{if } \varphi = \varphi_1 \wedge \varphi_2 \\ \mathcal{M} \models_I \varphi_1 \text{ or } \mathcal{M} \models_I \varphi_2 & \text{if } \varphi = \varphi_1 \vee \varphi_2 \\ \mathcal{M} \models_{I[x \mapsto a]} \psi \text{ for all } a \in A & \text{if } \varphi = \forall x.\, \psi \\ \mathcal{M} \models_{I[x \mapsto a]} \psi \text{ for some } a \in A & \text{if } \varphi = \exists x.\, \psi \end{cases}$$

Example

function and predicate symbols $\mathcal{F} = \{\mathsf{f}/1, \mathsf{a}/0\}$ and $\mathcal{P} = \{\mathsf{R}/2\}$

1. model $\mathcal{M}_1$: universe $A_1 = \mathbb{N}$

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- formula φ without free variables is called sentence
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- φ entails ψ in T (denoted $\varphi \models_T \psi$) if $\varphi \wedge \neg\psi$ is T-unsatisfiable

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- $M = l_1, \dots, l_k$ is T-satisfiable if $l_1 \wedge \cdots \wedge l_k$ is
- M is T-model of φ if $M \models \varphi$ and M is T-satisfiable
- φ entails ψ in T (denoted $\varphi \models_T \psi$) if $\varphi \wedge \neg\psi$ is T-unsatisfiable
- φ and ψ are T-equivalent (denoted $\varphi \equiv_T \psi$) if $\varphi \models_T \psi$ and $\psi \models_T \varphi$

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Example

- $x = y \wedge y \neq z$

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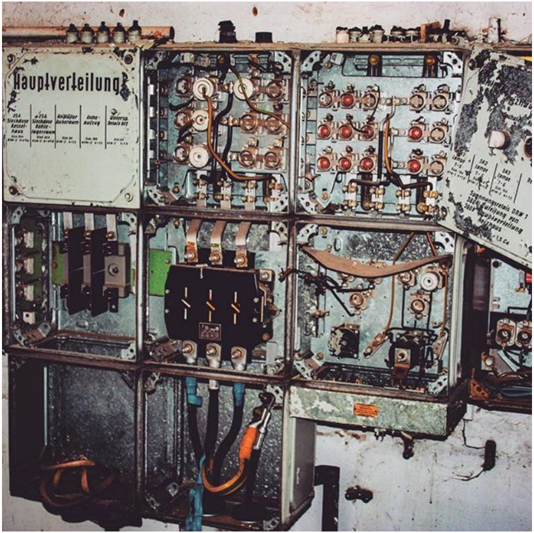
- $\mathsf{P}(\mathsf{a}) \wedge \mathsf{a} \neq \mathsf{b} \models_{\mathsf{EQ}} \neg\mathsf{P}(\mathsf{b})$ ✗

Uninterpreted Functions in Real Life



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The Eager Paradigm

Aim

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- requires sophisticated translation for each theory:
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- advantage: use SAT solver off the shelf
- drawbacks:
  - expensive translations: infeasible for large formulas
  - even more complicated with multiple theories

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     - otherwise T-solver generates T-consequence C of φ excluding v, repeat from 1 with $\varphi \wedge C$

Example

$$g(a) = c \land (\neg(f(g(a)) = f(c)) \lor g(a) = d) \land \neg(c = d\ )$$

Example

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2. unsatisfiable

Outline

- Summary of Last Week
- Satisfiability Modulo Theories
- DPLL(T)
- Using SMT Solvers with Theories

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Definition (DPLL(T) Transition Rules)

DPLL(T) consists of DPLL rules unit propagate, decide, fail, and restart plus

Approach

- most state-of-the-art SMT solvers use DPLL(T):
  lazy approach combining DPLL with theory propagation
- advantages: not specific to theory, also extends to theory combinations

Definition (DPLL(T) Transition Rules)

DPLL(T) consists of DPLL rules unit propagate, decide, fail, and restart plus

- T-backjump $\quad M\, l^d\, N \parallel F, C \quad \Longrightarrow \quad M\, l' \parallel F, C$
  if $M\, l^d\, N \models \neg C$ and $\exists$ clause $C' \vee l'$ such that
  - $F, C \models_T C' \vee l'$
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  if $M \vDash_T l$, literal l or l^c occurs in F, and l is undefined in M

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- T-solver checks T-satisfiability of model M whenever literal is added to M

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Remark

all three improvements can be combined

Example (Revisited with DPLL(T))

$$\underbrace{\mathsf{g}(\mathsf{a}) = \mathsf{c}}_{1} \wedge (\neg(\underbrace{\mathsf{f}(\mathsf{g}(\mathsf{a})) = \mathsf{f}(\mathsf{c})}_{2}) \vee \underbrace{\mathsf{g}(\mathsf{a}) = \mathsf{d}}_{3}) \wedge \neg(\underbrace{\mathsf{c} = \mathsf{d}}_{4})$$

$$\| \; 1, \; (\overline{2} \vee 3), \; \overline{4}$$

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$$\Longrightarrow \quad 1 \, \| \; 1, \; (\overline{2} \vee 3), \; \overline{4} \qquad \text{unit propagate}$$

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	$\parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	
$\Longrightarrow$	$1 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	unit propagate
$\Longrightarrow$	$1\ \overline{4} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	unit propagate

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$$
\begin{array}{lrl}
 & \| \; 1,\ (\overline{2} \vee 3),\ \overline{4} & \\
\Longrightarrow & 1 \; \| \; 1,\ (\overline{2} \vee 3),\ \overline{4} & \text{unit propagate} \\
\Longrightarrow & 1\,\overline{4} \; \| \; 1,\ (\overline{2} \vee 3),\ \overline{4} & \text{unit propagate} \\
\Longrightarrow & 1\,\overline{4}\,\overline{2}^{d} \; \| \; 1,\ (\overline{2} \vee 3),\ \overline{4} & \text{decide}
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$\Longrightarrow$	$1\,\overline{4}\,\overline{2}^{d} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2)$	T-learn
$\Longrightarrow$	$1\,\overline{4}\,2 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2)$	T-backjump
$\Longrightarrow$	$1\,\overline{4}\,2\,3 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2)$	unit propagate

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\Longrightarrow & \text{FailState} & & \text{fail}
\end{array}
$$

Lazyness in DPLL(T)


I'D LIKE TO HIRE SOMEONE TO DO ALL OF THE UNPLEASANT PARTS OF MY JOB.
THAT WAY I'LL BE FREE TO CONCENTRATE ON STRATEGY.
FAIR ENOUGH.
SECONDLY, I'D LIKE TO HIRE SOMEONE TO DO STRATEGY.

T-solver

Lazyness in DPLL(T)

T-solver SAT solver

Outline

- Summary of Last Week
- Satisfiability Modulo Theories
- DPLL(T)
- Using SMT Solvers with Theories

Example (SMT-LIB 2 for Propositional Logic)

formula $(x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee \neg x_1) \wedge (\neg x_1 \vee x_2 \vee x_3)$ can be expressed by

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(declare-const x3 Bool)
(assert (or x1 (not x3)))
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(check-sat)
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- `get-model` prints model (after satisfiability check)

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$f(f(a)) = a \land f(a) = b \land \neg(a = b)$ is expressed as

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(declare-sort A)
(declare-const a A)
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(declare-fun f (A) A)
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- prefix notation as in `(f (f a))` to denote $f(f(a))$ and `(=` x y`)` for equality
- `(distinct` x y`)` is equivalent to `not(=` x y`)`

Example (SMT-LIB 2 for LIA)

$2x \geqslant y + z \wedge \neg(x = y)$ is expressed as

```
(declare-const x Int)
(declare-const y Int)
(declare-const z Int)
(assert (>= (* 2 x) (+ y z)))
(assert (not (= x y)))
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Integer Arithmetic in SMT-LIB 2

- `declare-const` x `Int` creates integer variable named x

Example (SMT-LIB 2 for LIA)

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(declare-const x Int)
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Example (SMT-LIB 2 for LIA)

$2x \geqslant y + z \wedge \neg(x = y)$ is expressed as

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- `=` also covers equality on $\mathbb{Z}$
- `<`, `<=`, `>`, `>=` are $<_{\mathbb{Z}}$, $\leqslant_{\mathbb{Z}}$, $>_{\mathbb{Z}}$, $\geqslant_{\mathbb{Z}}$

EUF in python/z3

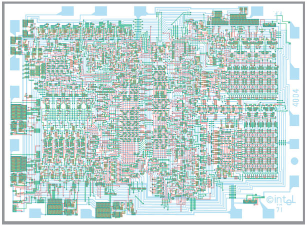
```
A = DeclareSort('A') # new uninterpreted sort named 'A'
a = Const('a', A) # create constant of sort A
b = Const('b', A) # create another constant of sort A
f = Function('f', A, A) # create function of sort A -> A

s = Solver()
s.add(f(f(a)) == a, f(a) == b, a != b)

print(s.check()) # sat
m = s.model()
print("interpretation assigned to A:")
print(m[A]) # [A!val!0, A!val!1]
print("interpretations:")
print(m[f]) # [A!val!0 -> A!val!1, A!val!1 -> A!val!0, ...]
print(m[a]) # A!val!0
print(m[b]) # A!val!1
```

EUF Application: Verification of Microprocessors

- verify that 3-stage pipelined MIPS processor satisfies intended instruction set architecture



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  - computation logic as UF
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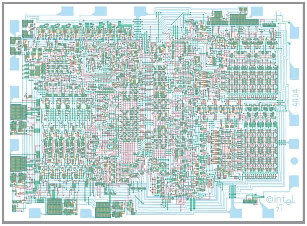

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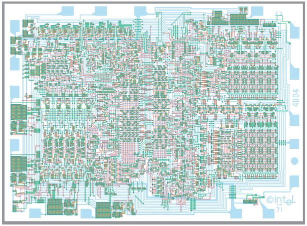
- EUF ensures functional consistency: same data results in same computation



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Journal of the ACM 53(6), pp. 937–977, 2006.

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