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SAT and SMT Solving

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Outline

- Summary of Last Week
- Satisfiability Modulo Theories
- DPLL(T)
- Using SMT Solvers with Theories

Definitions

for unsatisfiable CNF formula φ given as set of clauses

- $\psi \subseteq \varphi$ such that $\bigwedge_{C \in \psi} C$ is unsatisfiable is unsatisfiable core (UC) of φ
- minimal unsatisfiable core ψ is UC such that every subset of ψ is satisfiable
- SUC (minimum unsatisfiable core) is UC such that $|\psi|$ is minimal

Remark

SUC is always minimal unsatisfiable core

Definition (Resolution Graph)

directed acyclic graph $G = (V, E)$ is resolution graph for set of clauses φ if

1. $V = V_i \uplus V_c$ is set of clauses and $V_i = \varphi$,
2. V_i nodes have no incoming edges,
3. there is exactly one node $\Box$ without outgoing edges,
4. $\forall C \in V_c \; \exists$ edges $D \to C$, $D' \to C$ such that C is resolvent of D and D', and
5. there are no other edges.

Algorithm minUnsatCore(φ)

Input: unsatisfiable formula φ
Output: minimal unsatisfiable core of φ

build resolution graph $G = (V_i \uplus V_c, E)$ for φ
while $\exists$ unmarked clause in V_i **do**
  $C \leftarrow$ unmarked clause in V_i
  if SAT($\overline{Reach_G(C)}$) **then** ▷ subgraph without C satisfiable?
    mark C ▷ C is UC member
  else
    build resolution graph $G' = (V_i' \uplus V_c', E')$ for $\overline{Reach_G(C)}$
    $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
    $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
    $G \leftarrow (V_i \cup V_c, E)$
    $G \leftarrow G|_{BReach_G(\square)}$ ▷ restrict to nodes with path to $\square$
return V_i

Theorem

if φ unsatisfiable then minUnsatCore(φ) is minimal unsatisfiable core of φ

Definition (Partial minUNSAT)

pminUNSAT(χ, φ) is minimal $|\psi|$ such that $\psi \subseteq \varphi$ and $\chi \wedge \bigwedge_{C \in \psi} \neg C$ satisfiable

Algorithm FuMalik(χ, φ)

Input: clause set φ and satisfiable clause set χ

```
cost ← 0
while ¬SAT(χ ∪ φ) do
    UC ← unsatCore(χ ∪ φ)                          ▷ must be minimal
    B ← ∅
    for C ∈ UC ∩ φ do                               ▷ loop over soft clauses in core
        b ← new blocking variable
        φ ← φ \ {C} ∪ {C ∨ b}
        B ← B ∪ {b}
    χ ← χ ∪ CNF(∑_{b∈B} b = 1)                     ▷ cardinality constraint is hard
    cost ← cost + 1
return cost
```

Theorem

FuMalik(χ, φ) = pminUNSAT(χ, φ)

$|\varphi|$ = pminUNSAT(χ, φ) + pmaxSAT(χ, φ)

Outline

- Summary of Last Week
- Satisfiability Modulo Theories
  - Recap: First-Order Logic
  - Eager and Lazy Paradigms
- DPLL(T)
- Using SMT Solvers with Theories

SMT Solving

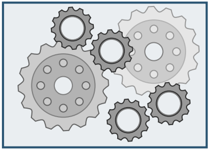
input: formula φ involving theory T

output: SAT + valuation v such that $v(\varphi) = T$ if φ is T-satisfiable

UNSAT otherwise

φ $\longrightarrow$

$a + b \geqslant c \vee (a = 0 \wedge p)$



SMT solver

SAT (I) $v(a) = 3$ $v(b) = 0$ $v(c) = 0$ $v(p) = \mathsf{T}$

UNSAT

Example (Common theories)

- arithmetic $2a + b \geqslant c \vee (a - b = c + 3 \wedge p)$
- uninterpreted functions $\mathsf{f}(x, y) \neq \mathsf{f}(y, x) \wedge \mathsf{g}(\mathsf{a}) = \mathsf{a} \rightarrow \mathsf{g}(\mathsf{f}(x, x)) = \mathsf{g}(y)$
- bit vectors $((\mathsf{zext}_{32}\ a_8) + b_{32}) \times c_{32} >_u 0_{32}$

First-Order Logic: Syntax

Definitions (Signature)

- signature $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ consists of
  - set of function symbols $\mathcal{F}$
  - set of predicate symbols $\mathcal{P}$

  where each symbol is associated with fixed arity (i.e., number of arguments)
- function/predicate symbols with arity
  - 1 are called unary
  - 2 are called binary
  - 0 are called constants

Definitions (Formulas)

- Σ-terms t are built according to grammar

  $$t \quad ::= \quad x \mid c \mid f(\underbrace{t, \dots, t}_{n})$$

  infinite set of variables X

  for constant $c \in \mathcal{F}$, function symbol $f \in \mathcal{F}$ of arity $n > 0$, and variable $x \in X$
- Σ-formulas are built according to grammar

  $$\varphi \quad ::= \quad Q \mid P(\underbrace{t, \dots, t}_{n}) \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x.\, \varphi \mid \exists x.\, \varphi$$

  for constant $Q \in \mathcal{P}$, predicate symbol $P \in \mathcal{P}$ of arity $n > 0$, and Σ-terms t
- variable x is free in φ if it is not bound by quantifier above

Notation

write f/n or P/n to express that f or P have arity n

Example

- let $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{F} := \{\mathrm{a}/0,\ \mathrm{b}/0,\ \mathrm{f}/1,\ \mathrm{g}/2\}$ and $\mathcal{P} := \{\mathrm{Q}/0,\ \mathrm{P}/1,\ =/2\}$
- the following are Σ-terms:
  - a, b, and $\mathrm{f}(\mathrm{a})$
  - x, y, and z — x, y, z are variables
  - $\mathrm{g}(\mathrm{a}, \mathrm{f}(x))$ and $\mathrm{g}(\mathrm{g}(\mathrm{a}, y), \mathrm{f}(\mathrm{b}))$
- the following are Σ-formulas (free variables highlighted):
  - $\mathrm{a} = \mathrm{f}(\mathrm{b})$
  - $\mathrm{P}(\mathrm{a}) \wedge \mathrm{Q}$
  - $\neg(\mathrm{P}(\mathrm{a}) \wedge \mathrm{P}(x) \wedge \mathrm{P}(y))$
  - $\exists x.\, \mathrm{P}(x)$
  - $\mathrm{P}(x) \vee (\exists x.\, \mathrm{P}(x) \wedge \mathrm{f}(y) = x)$
  - $\forall x\, y\, z.\ (x = y \wedge y = z \rightarrow x = z)$
  - $\forall x\, y.\ (x = y \rightarrow y = x)$

write $\varphi \rightarrow \psi$ for $\neg\varphi \vee \psi$

First-Order Logic: Semantics

Definition (Model)

model $\mathcal{M}$ for signature $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ consists of

1. non-empty set A (universe of concrete values)
2. function $f^{\mathcal{M}} \colon A^n \to A$ for every n-ary $f \in \mathcal{F}$
3. set of n-tuples $P^{\mathcal{M}} \subseteq A^n$ for every n-ary $P \in \mathcal{P}$

Example

function and predicate symbols $\mathcal{F} = \{\mathsf{f}/1, \mathsf{a}/0\}$ and $\mathcal{P} = \{\mathsf{R}/2\}$

1. model $\mathcal{M}_1$: universe $A_1 = \mathbb{N}$
   $\mathsf{f}^{\mathcal{M}_1}(x) = 2x + 1$
   $\mathsf{a}^{\mathcal{M}_1} = 0$
   $\mathsf{R}^{\mathcal{M}_1} = \{(x, y) \mid x < y\}$
2. model $\mathcal{M}_2$: universe A_2 is set of all Twitter users
   $\mathsf{f}^{\mathcal{M}_2}(x) =$ last person who started following x (or x if no follower)
   $\mathsf{a}^{\mathcal{M}_2} =$ `@elonmusk`
   $\mathsf{R}^{\mathcal{M}_2} = \{(x, y) \mid x \text{ follows } y\}$

Definitions

- environment for model $\mathcal{M} = \langle A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}} \rangle$ is mapping $I \colon X \to A$
- value $t^{\mathcal{M},I}$ of term t in model $\mathcal{M}$ wrt environment I is defined inductively:

$$t^{\mathcal{M},I} = \begin{cases} I(t) & \text{if } t \text{ is a variable} \\ f^{\mathcal{M}}(t_n^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

- for environment I, variable x and $a \in A$, extended environment $I[x \mapsto a]$ is

$$(I[x \mapsto a])(y) = \begin{cases} a & \text{if } x = y \\ I(y) & \text{otherwise} \end{cases}$$

- satisfaction relation $\mathcal{M} \vDash_I \varphi$ is defined inductively:

$$\mathcal{M} \vDash_I \varphi \quad \Longleftrightarrow \quad \begin{cases} (t_n^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I}) \in P^{\mathcal{M}} & \text{if } \varphi = P(t_1, \dots, t_n) \\ \mathcal{M} \nvDash_I \psi & \text{if } \varphi = \neg\psi \\ \mathcal{M} \vDash_I \varphi_1 \text{ and } \mathcal{M} \vDash_I \varphi_2 & \text{if } \varphi = \varphi_1 \wedge \varphi_2 \\ \mathcal{M} \vDash_I \varphi_1 \text{ or } \mathcal{M} \vDash_I \varphi_2 & \text{if } \varphi = \varphi_1 \vee \varphi_2 \\ \mathcal{M} \vDash_{I[x \mapsto a]} \psi \text{ for all } a \in A & \text{if } \varphi = \forall x.\, \psi \\ \mathcal{M} \vDash_{I[x \mapsto a]} \psi \text{ for some } a \in A & \text{if } \varphi = \exists x.\, \psi \end{cases}$$

Example

function and predicate symbols $\mathcal{F} = \{\mathsf{f}/1, \mathsf{a}/0\}$ and $\mathcal{P} = \{\mathsf{R}/2\}$

1. model $\mathcal{M}_1$: universe $A_1 = \mathbb{N}$
   $\mathsf{f}^{\mathcal{M}_1}(x) = 2x + 1$
   $\mathsf{a}^{\mathcal{M}_1} = 0$
   $\mathsf{R}^{\mathcal{M}_1} = \{(x, y) \mid x < y\}$
2. model $\mathcal{M}_2$: universe A_2 is set of all Twitter users
   $\mathsf{f}^{\mathcal{M}_2}(x) =$ last person who started following x (or x if no follower)
   $\mathsf{a}^{\mathcal{M}_2} =$ `@elonmusk`
   $\mathsf{R}^{\mathcal{M}_2} = \{(x, y) \mid x \text{ follows } y\}$
3. model $\mathcal{M}_3$: universe A_3 is set of all days since year 2000
   $\mathsf{f}^{\mathcal{M}_3}(x)$ is day after x
   $\mathsf{a}^{\mathcal{M}_3} =$ "11.09.2001"
   $\mathsf{R}^{\mathcal{M}_3} = \{(x, y) \mid y \text{ is after } x\}$

$\varphi_1 = \exists x.\mathsf{R}(x, a)$ $\quad$ $\varphi_2 = \forall x.\mathsf{R}(x, \mathsf{f}(x))$ $\quad$ $\varphi_3 = \forall x\, y\, z.\, \mathsf{R}(x, y) \wedge \mathsf{R}(y, z) \rightarrow \mathsf{R}(x, z)$

$\mathcal{M}_1 \not\models_I \varphi_1$	$\mathcal{M}_1 \models_I \varphi_2$	$\mathcal{M}_1 \models_I \varphi_3$
$\mathcal{M}_2 \models_I \varphi_1$	$\mathcal{M}_2 \not\models_I \varphi_2$	$\mathcal{M}_2 \not\models_I \varphi_3$
$\mathcal{M}_3 \models_I \varphi_1$	$\mathcal{M}_3 \models_I \varphi_2$	$\mathcal{M}_3 \models_I \varphi_3$

Remark

- formula φ without free variables is called sentence
- if φ is sentence, $\mathcal{M} \models_I \varphi$ is independent of I, so simply write $\mathcal{M} \models \varphi$

Definition

- formula φ is satisfiable if $\mathcal{M} \models_I \varphi$ for some $\mathcal{M}$ and I
- set of formulas T is satisfiable if $\mathcal{M} \models_I \bigwedge_{\varphi \in T} \varphi$ for some $\mathcal{M}$ and I

Definition (Theory)

Σ-theory T is set of Σ-sentences that is satisfiable

Definitions

for Σ-theory T, Σ-formulas φ and ψ and list of literals M:

- φ is T-satisfiable (or T-consistent) if $\varphi \cup \{T\}$ is satisfiable
- φ is T-unsatisfiable if not T-satisfiable
- $M = l_1, \dots, l_k$ is T-satisfiable if $l_1 \wedge \cdots \wedge l_k$ is
- M is T-model of φ if $M \models \varphi$ and M is T-satisfiable
- φ entails ψ in T (denoted $\varphi \models_T \psi$) if $\varphi \wedge \neg\psi$ is T-unsatisfiable
- φ and ψ are T-equivalent (denoted $\varphi \equiv_T \psi$) if $\varphi \models_T \psi$ and $\psi \models_T \varphi$

Definition (Theory of Equality EQ)

- signature: no function symbols, binary predicate $=$
- axioms:
  - $\forall x.\ (x = x)$
  - $\forall x\, y.\ (x = y \ \rightarrow\ y = x)$
  - $\forall x\, y\, z.\ (x = y \wedge y = z \ \rightarrow\ x = z)$

Example

- $x = y \wedge y \neq z$ EQ-satisfiable
- $x = y \wedge y \neq z \wedge (z = x \vee x = z)$ EQ-unsatisfiable
- $x = y \wedge y \neq z \models_{\mathsf{EQ}} z \neq x$ ✓
- $x = y \equiv_{\mathsf{EQ}} y = x$ ✓
- $x = y \wedge y \neq z \equiv_{\mathsf{EQ}} z \neq x$ ✗

Definition (Theory of Equality With Uninterpreted Functions EUF)

- signature: function symbols $\mathcal{F}$, predicate symbols $\mathcal{P}$ including binary $=$
- axioms:

$$\forall x.\ (x = x) \quad \forall x\ y.\ (x = y \ \rightarrow\ y = x) \quad \forall x\ y\ z.\ (x = y \wedge y = z \ \rightarrow\ x = z)$$

plus for all n-ary $f \in \mathcal{F}$:

$$\forall x_1\ y_1\ \dots\ x_n\ y_n.\ (x_1 = y_1 \wedge \cdots \wedge x_n = y_n \ \rightarrow\ f(x_1, \dots, x_n) = f(y_1, \dots, y_n))$$

plus for all n-ary $P \in \mathcal{P} \setminus \{=\}$:

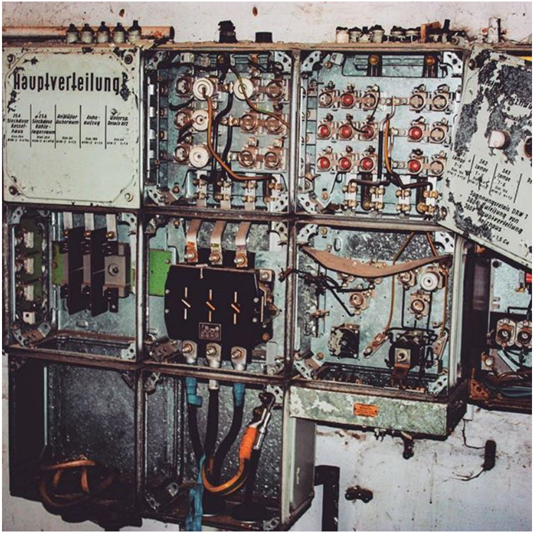
$$\forall x_1\ y_1\ \dots\ x_n\ y_n.\ (x_1 = y_1 \wedge \cdots \wedge x_n = y_n \ \rightarrow\ (P(x_1, \dots, x_n) \rightarrow P(y_1, \dots, y_n)))$$

Example

for $\mathcal{F} = \{\mathsf{a}/0,\ \mathsf{b}/0,\ \mathsf{f}/1, \mathsf{g}/2\}$ and $\mathcal{P} = \{=/2, Q/1\}$

- $\mathsf{a} = \mathsf{b} \wedge \mathsf{f}(\mathsf{a}) = \mathsf{a} \wedge \mathsf{g}(\mathsf{f}(\mathsf{a}), \mathsf{b}) \neq \mathsf{g}(\mathsf{b}, \mathsf{b})$ UEQ-unsatisfiable
- $\mathsf{a} = \mathsf{b} \wedge \mathsf{f}(\mathsf{a}) \neq \mathsf{b} \wedge \mathsf{g}(\mathsf{g}(\mathsf{b}, \mathsf{b}), \mathsf{b}) = \mathsf{g}(\mathsf{b}, \mathsf{b})$ UEQ-satisfiable
- $\mathsf{a} = \mathsf{b} \wedge \mathsf{f}(\mathsf{a}) \neq \mathsf{a} \models_{\mathsf{UEQ}} \mathsf{f}(\mathsf{a}) \neq \mathsf{b}$ ✓
- $\mathsf{f}(\mathsf{a}) = \mathsf{a} \wedge \mathsf{P}(\mathsf{a}) \equiv_{\mathsf{UEQ}} \mathsf{f}(\mathsf{a}) = \mathsf{a} \wedge \mathsf{P}(\mathsf{f}(\mathsf{a}))$ ✓
- $\mathsf{P}(\mathsf{a}) \wedge \mathsf{a} \neq \mathsf{b} \models_{\mathsf{EQ}} \neg\mathsf{P}(\mathsf{b})$ ✗

Uninterpreted Functions in Real Life



Theories of Interest in SMT Solvers

- equality + uninterpreted functions (EUF) $f(x, a) = g(y)$
- difference logic (DL) $x - y \leqslant 1$
- linear arithmetic $3x - 5y + 7z \leqslant 1$
  - over integers $\mathbb{Z}$ (LIA)
  - over reals $\mathbb{R}$ (LRA)
- arrays (A) $\mathrm{read}(\mathrm{write}(A, i, v), j)$
- bitvectors (BV) $((\mathrm{zext}_{32}\ a_8) + b_{32}) \times c_{32} >_u 0_{32}$
- strings $x \mathbin{@} y = z \mathbin{@} \mathrm{replace}(y, \mathtt{a}, \mathtt{b})$
- ...
- their combinations

SMT-LIB

- language standard and benchmarks: `http://www.smt-lib.org`
- annual solver competition: `http://www.smt-comp.org`
- solvers: Yices, OpenSMT, MathSAT, Z3, CVC4, Barcelogic, ...

The Eager Paradigm

Aim

given Σ-theory T and Σ-formula φ mixing propositional logic with symbols from Σ, determine T-satisfiability

Approach 1: Eager SMT Solving

- use satisfiability-preserving transformation from T literals to SAT formula, ship one big formula to SAT solver
- requires sophisticated translation for each theory:
  done for EUF, difference logic, linear integer arithmetic, arrays
- still dominant approach for bit-vector arithmetic (known as "bit blasting")
- advantage: use SAT solver off the shelf
- drawbacks:
  - expensive translations: infeasible for large formulas
  - even more complicated with multiple theories

The Lazy Paradigm

Aim

given Σ-theory T and Σ-formula φ mixing propositional logic with symbols from Σ, determine T-satisfiability

Idea

use specialized T-solver that can deal with conjunction of theory literals

Approach 2: Lazy SMT Solving

1. abstract φ to propositional CNF:
   - "forget theory" by replacing T-literals with fresh propositional variables
   - obtain pure SAT formula, transform to CNF formula ψ
2. ship ψ to SAT solver
   - if ψ unsatisfiable, so is φ
   - if ψ satisfiable by v, check v with T-solver:
     - if v is T-consistent then also φ is satisfiable
     - otherwise T-solver generates T-consequence C of φ excluding v, repeat from 1 with $\varphi \wedge C$

Example

$$g(a) = c \land (\neg(f(g(a)) = f(c)) \lor g(a) = d) \land \neg(c = d\ )$$

1. abstract to propositional skeleton $\psi_1 = x_1 \land (\neg x_2 \lor x_3) \land \neg x_4$
2. satisfiable: $v_1(x_1) = \mathrm{T}$ and $v_1(x_2) = v_1(x_4) = \mathrm{F}$
   - T-solver gets $g(a) = c \land f(g(a)) \neq f(c) \land c \neq d$
   - T-unsatisfiable: $g(a) = c$ implies $f(g(a)) = f(c)$
   - block valuation v_1 in future: add $\neg x_1 \lor x_2$

1. $\psi_2 = x_1 \land (\neg x_2 \lor x_3) \land \neg x_4 \land (\neg x_1 \lor x_2)$
2. satisfiable: $v_2(x_1) = v_2(x_2) = v_2(x_3) = \mathrm{T}$ and $v_2(x_4) = \mathrm{F}$
   - T-solver gets $g(a) = c \land f(g(a)) = f(c) \land g(a) = d \land c \neq d$
   - T-unsatisfiable
   - block valuation v_2 in future: add $\neg x_1 \lor \neg x_3 \lor x_4$

1. $\psi_3 = x_1 \land (\neg x_2 \lor x_3) \land \neg x_4 \land (\neg x_1 \lor x_2 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor x_4)$
2. unsatisfiable

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- DPLL(T)
- Using SMT Solvers with Theories

Approach

- most state-of-the-art SMT solvers use DPLL(T): lazy approach combining DPLL with theory propagation
- advantages: not specific to theory, also extends to theory combinations

Definition (DPLL(T) Transition Rules)

DPLL(T) consists of DPLL rules unit propagate, decide, fail, and restart plus

- T-backjump $\quad M\ l^d\ N \parallel F, C \quad \Longrightarrow \quad M\ l' \parallel F, C$
  if $M\ l^d\ N \vDash \neg C$ and $\exists$ clause $C' \vee l'$ such that
  - $F, C \vDash_T C' \vee l'$
  - $M \vDash \neg C'$ and l' is undefined in M, and l' or l'^c occurs in F or in $M\ l^d\ N$
- T-learn $\quad M \parallel F \quad \Longrightarrow \quad M \parallel F,\ C$
  if $F \vDash_T C$ and all atoms of C occur in M or F
- T-forget $\quad M \parallel F,\ C \quad \Longrightarrow \quad M \parallel F$
  if $F \vDash_T C$
- T-propagate $\quad M \parallel F \quad \Longrightarrow \quad M\ l \parallel F$
  if $M \vDash_T l$, literal l or l^c occurs in F, and l is undefined in M

Simple Strategy using DPLL(T)

- whenever state $M \parallel F$ is final wrt unit propagate, decide, fail, T-backjump: check T-satisfiability of M with T-solver
- if M is T-consistent then T-satisfiability is proven
- otherwise $\exists l_1, \dots, l_k$ subset of M such that $F \vDash_T \neg(l_1 \wedge \cdots \wedge l_k)$
- use T-learn to add $\neg l_1 \vee \cdots \vee \neg l_k$
- apply restart

Improvement 1: Incremental T-Solver

- T-solver checks T-satisfiability of model M whenever literal is added to M

Improvement 2: On-Line SAT solver

- after T-learn added clause, apply fail or T-backjump instead of restart

Improvement 3: Eager Theory Propagation

- apply T-propagate before decide

Remark

all three improvements can be combined

Example (Revisited with DPLL(T))

$$\underbrace{g(a) = c}_{1} \wedge (\neg(\underbrace{f(g(a)) = f(c)}_{2}) \vee \underbrace{g(a) = d}_{3}) \wedge \neg(\underbrace{c = d}_{4})$$

	$\parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	
$\Longrightarrow$	$1 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	unit propagate
$\Longrightarrow$	$1\ \overline{4} \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	unit propagate
$\Longrightarrow$	$1\ \overline{4}\ \overline{2}^d \parallel 1,\ (\overline{2} \vee 3),\ \overline{4}$	decide
$\Longrightarrow$	$1\ \overline{4}\ \overline{2}^d \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2)$	T-learn
$\Longrightarrow$	$1\ \overline{4}\ 2 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2)$	T-backjump
$\Longrightarrow$	$1\ \overline{4}\ 2\ 3 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2)$	unit propagate
$\Longrightarrow$	$1\ \overline{4}\ 2\ 3 \parallel 1,\ (\overline{2} \vee 3),\ \overline{4},\ (\overline{1} \vee 2),\ (\overline{1} \vee \overline{3} \vee 4)$	T-learn
$\Longrightarrow$	FailState	fail

Lazyness in DPLL(T)

T-solver

SAT solver

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Example (SMT-LIB 2 for Propositional Logic)

formula $(x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee \neg x_1) \wedge (\neg x_1 \vee x_2 \vee x_3)$ can be expressed by

```
(declare-const x1 Bool)
(declare-const x2 Bool)
(declare-const x3 Bool)
(assert (or x1 (not x3)))
(assert (or x2 x3 (not x1)))
(assert (or (not x1) x2 x3))
(check-sat)
(get-model)
```

Propositional Logic in SMT-LIB 2

- `declare-const` *x* `Bool` creates propositional variable named *x*
- prefix notation for `and`, `or`, `not`, `implies`
- `assert` demands given formula to be satisfied
- `check-sat` issues satisfiability check of conjunction of assertions
- `get-model` prints model (after satisfiability check)

Example (SMT-LIB 2 for EUF)

$f(f(a)) = a \wedge f(a) = b \wedge \neg(a = b)$ is expressed as

```
(declare-sort A)
(declare-const a A)
(declare-const b A)
(declare-fun f (A) A)
(assert (= (f (f a)) a))
(assert (= (f a) b))
(assert (distinct a b))
(check-sat)
(get-model)
```

EUF in SMT-LIB 2

- terms must have sort, so declare fresh sort and use for all symbols: `declare-sort` S creates sort named S
- `declare-const` x s creates variable named x of sort S
- `declare-fun` F $(S_1 \dots S_n)$ T creates uninterpreted $F \colon S_1 \times \cdots \times S_n \to T$
- prefix notation as in `(f (f a))` to denote $f(f(a))$ and `(=` x y`)` for equality
- `(distinct` x y`)` is equivalent to `not(=` x y`)`

Example (SMT-LIB 2 for LIA)

$2x \geqslant y + z \land \neg(x = y)$ is expressed as

```
(declare-const x Int)
(declare-const y Int)
(declare-const z Int)
(assert (>= (* 2 x) (+ y z)))
(assert (not (= x y)))
(check-sat)
(get-model)
```

Integer Arithmetic in SMT-LIB 2

- `declare-const` x `Int` creates integer variable named x
- numbers `0`, `1`, `-1`, `42`,... are built-in
- `+`, `*`, `-` are $+_{\mathbb{Z}}$, $\cdot_{\mathbb{Z}}$, $-_{\mathbb{Z}}$, used in prefix notation: `(+ 2 3)`
- `=` also covers equality on $\mathbb{Z}$
- `<`, `<=`, `>`, `>=` are $<_{\mathbb{Z}}$, $\leqslant_{\mathbb{Z}}$, $>_{\mathbb{Z}}$, $\geqslant_{\mathbb{Z}}$

EUF in python/z3

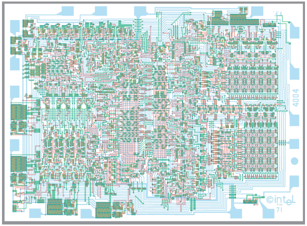
```
A = DeclareSort('A') # new uninterpreted sort named 'A'
a = Const('a', A) # create constant of sort A
b = Const('b', A) # create another constant of sort A
f = Function('f', A, A) # create function of sort A -> A

s = Solver()
s.add(f(f(a)) == a, f(a) == b, a != b)

print(s.check()) # sat
m = s.model()
print("interpretation assigned to A:")
print(m[A]) # [A!val!0, A!val!1]
print("interpretations:")
print(m[f]) # [A!val!0 -> A!val!1, A!val!1 -> A!val!0, ...]
print(m[a]) # A!val!0
print(m[b]) # A!val!1
```

EUF Application: Verification of Microprocessors

- verify that 3-stage pipelined MIPS processor satisfies intended instruction set architecture
- encoding
  - data as bit sequence
  - memory as uninterpreted function (UF)
  - computation logic as UF
  - control logic as uninterpreted predicate
- EUF ensures functional consistency: same data results in same computation



Miroslav N. Velev and Randal E. Bryant.
Bit-level abstraction in the verification of pipelined microprocessors by correspondence checking.
In Proc. of Formal Methods in Computer-Aided Design, pp. 18–35, 1998.

DPLL(T)

Robert Nieuwenhuis, Albert Oliveras, and Cesare Tinelli.
Solving SAT and SAT Modulo Theories: From an Abstract Davis-Putnam-Logemann-Loveland Procedure to DPLL(T).
Journal of the ACM 53(6), pp. 937–977, 2006.

Application

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