SAT and SMT Solving

Sarah Winkler

KRDB
Department of Computer Science
Free University of Bozen-Bolzano

lecture 6
WS 2022

Outline

- Summary of Last Week
- Deciding EQ: Equality Graphs
- Deciding EUF: Congruence Closure
- Correctness of DPLL(T)
- Some More Practical SMT

First-Order Logic: Syntax

Definitions

- signature $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ consists of
  - set of function symbols $\mathcal{F}$
  - set of predicate symbols $\mathcal{P}$

  where each symbol is associated with fixed arity
- Σ-terms t are built according to grammar

$$t \quad ::= \quad x \mid c \mid f(\underbrace{t, \dots, t}_{n})$$

- Σ-formulas φ are built according to grammar

$$\varphi \quad ::= \quad Q \mid P(\underbrace{t, \dots, t}_{n}) \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x.\, \varphi \mid \exists x.\, \varphi$$

- variable occurrence is free in φ if it is not bound by quantifier above
- formulas without free variables are sentences

First-Order Logic: Semantics

Definition

model $\mathcal{M}$ for signature $\Sigma = \langle \mathcal{F}, \mathcal{P} \rangle$ consists of

1. non-empty set A (universe of concrete values)
2. function $f^{\mathcal{M}} \colon A^n \to A$ for every n-ary $f \in \mathcal{F}$
3. set of n-tuples $P^{\mathcal{M}} \subseteq A^n$ for every n-ary $P \in \mathcal{P}$

Definitions

- environment for model $\mathcal{M}$ with universe A is mapping $I \colon X \to A$
- value $t^{\mathcal{M},I}$ of term t in model $\mathcal{M}$ wrt environment I:
  $t^{\mathcal{M},I} = I(t)$ if t is a variable, and $t^{\mathcal{M},I} = f^{\mathcal{M}}(t_n^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I})$ otherwise
-
$$\mathcal{M} \vDash_I \varphi \quad \Longleftrightarrow \quad \begin{cases} (t_n^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I}) \in P^{\mathcal{M}} & \text{if } \varphi = P(t_1, \dots, t_n) \\ \mathcal{M} \nvDash_I \psi & \text{if } \varphi = \neg\psi \\ \mathcal{M} \vDash_I \varphi_1 \text{ and } \mathcal{M} \vDash_I \varphi_2 & \text{if } \varphi = \varphi_1 \wedge \varphi_2 \\ \mathcal{M} \vDash_I \varphi_1 \text{ or } \mathcal{M} \vDash_I \varphi_2 & \text{if } \varphi = \varphi_1 \vee \varphi_2 \\ \mathcal{M} \vDash_{I[x \mapsto a]} \psi \text{ for all } a \in A & \text{if } \varphi = \forall x.\, \psi \\ \mathcal{M} \vDash_{I[x \mapsto a]} \psi \text{ for some } a \in A & \text{if } \varphi = \exists x.\, \psi \end{cases}$$

Definition

- formula φ is satisfiable if $\mathcal{M} \models_I \varphi$ for some $\mathcal{M}$ and I
- set of formulas T is satisfiable if $\mathcal{M} \models_I \bigwedge_{\varphi \in T} \varphi$ for some $\mathcal{M}$ and I

Remark

if φ is sentence, $\mathcal{M} \models_I \varphi$ is independent of I

Definition (Theory)

Σ-theory T is set of Σ-sentences that is satisfiable

Definitions

for theory T, formulas F and G and list of literals M:

- F is T-consistent (or T-satisfiable) if $\{F\} \cup T$ is satisfiable
- F is T-inconsistent (or T-unsatisfiable) if not T-consistent
- F entails G in T (denoted $F \vDash_T G$) if $F \wedge \neg G$ is T-inconsistent
- F and G are T-equivalent (denoted $F \equiv_T G$) if $F \vDash_T G$ and $G \vDash_T F$

Definition (Theory of Equality EQ)

- signature: no function symbols, binary predicate $=$
- axioms:

$$\forall x.\ (x = x) \quad \forall x\, y.\ (x = y \ \rightarrow\ y = x) \quad \forall x\, y\, z.\ (x = y \wedge y = z \ \rightarrow\ x = z)$$

Definition (Theory of Equality With Uninterpreted Functions EUF)

- signature: function symbols $\mathcal{F}$, predicate symbols $\mathcal{P}$ including binary $=$
- axioms:

$$\forall x.\ (x = x) \quad \forall x\, y.\ (x = y \ \rightarrow\ y = x) \quad \forall x\, y\, z.\ (x = y \wedge y = z \ \rightarrow\ x = z)$$

plus for all $f/n \in \mathcal{F}$ and $P/n \in \mathcal{P}$ functional consistency axioms:

$$\forall x_1 y_1 \ \ldots\ x_n y_n.\ (x_1 = y_1 \wedge \cdots \wedge x_n = y_n \ \rightarrow\ f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$$

$$\forall x_1\, y_1 \ \ldots\ x_n\, y_n.\ (x_1 = y_1 \wedge \cdots \wedge x_n = y_n \ \rightarrow\ (P(x_1, \ldots, x_n) \rightarrow P(y_1, \ldots, y_n)))$$

Definition

DPLL(T) consists of DPLL rules unit propagate, decide, fail, and restart plus

- T-backjump $\quad M\, l^d\, N \parallel F, C \quad \Longrightarrow \quad M\, l' \parallel F, C$

  if $M\, l^d\, N \models \neg C$ and $\exists$ clause $C' \vee l'$ such that
  - $F, C \models_T C' \vee l'$
  - $M \models \neg C'$ and l' is undefined in M, and l' or l'^c occurs in F or in $M\, l^d\, N$

- T-learn $\quad M \parallel F \quad \Longrightarrow \quad M \parallel F,\ C$

  if $F \models_T C$ and all atoms of C occur in M or F

- T-forget $\quad M \parallel F,\ C \quad \Longrightarrow \quad M \parallel F$

  if $F \models_T C$

- T-propagate $\quad M \parallel F \quad \Longrightarrow \quad M\, l \parallel F$

  if $M \models_T l$, literal l or l^c occurs in F, and l is undefined in M

Naive Lazy Approach in DPLL(T)

- whenever state $M \parallel F$ is final wrt unit propagate, decide, fail, T-backjump: check T-consistency of M with T-solver
- if M is T-consistent then satisfiability is proven
- otherwise $\exists l_1, \dots, l_k$ subset of M such that $\vDash_T \neg(l_1 \wedge \cdots \wedge l_k)$
- use T-learn to add $\neg l_1 \vee \cdots \vee \neg l_k$
- apply restart

Improvement 1: Incremental T-Solver

- T-solver checks T-consistency of model M whenever literal is added to M

Improvement 2: On-Line SAT solver

- after T-learn added clause, apply fail or T-backjump instead of restart

Improvement 3: Eager Theory Propagation

- apply T-propagate before decide

Outline

- Summary of Last Week
- Deciding EQ: Equality Graphs
- Deciding EUF: Congruence Closure
- Correctness of DPLL(T)
- Some More Practical SMT

Equality Graph

Aim

build theory solver for theory of equality (EQ)

Definition

- equality logic formula φ_{EQ} is set of equations and inequalities between variables
- write $\mathcal{V}ar(\varphi_{\text{EQ}})$ for set of variables occurring in φ_{EQ}

Definition

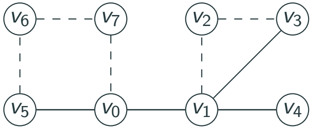
equality graph for φ_{EQ} is undirected graph $(V, E_=, E_{\neq})$ with two kinds of edges

- nodes $V = \mathcal{V}ar(\varphi_{\text{EQ}})$
- $(x, y) \in E_=$ iff $x = y$ in φ_{EQ} equality edge
- $(x, y) \in E_{\neq}$ iff $x \neq y$ in φ_{EQ} e

edges $E_=$ are drawn dashed, $E_{\neq}$ are drawn solid

Example

$v_0 \neq v_1 \quad v_0 \neq v_5 \quad v_1 = v_2 \quad v_1 \neq v_4 \quad v_1 \neq v_3 \quad v_2 = v_3 \quad v_5 = v_6 \quad v_6 = v_7 \quad v_7 = v_0$



Definition (Contradictory cycle)

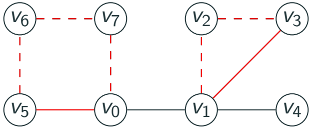
contradictory cycle is simple cycle in equality graph with one $E_{\neq}$ edge and all others $E_{=}$ edges

Theorem

φ_{EQ} is satisfiable iff its equality graph has no contradictory cycle

Example

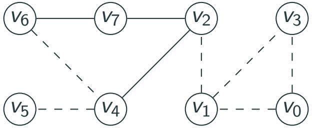
$v_0 \neq v_1 \quad v_0 \neq v_5 \quad v_1 = v_2 \quad v_1 \neq v_4 \quad v_1 \neq v_3 \quad v_2 = v_3 \quad v_5 = v_6 \quad v_6 = v_7 \quad v_7 = v_0$



unsatisfiable

Example

$v_0 = v_1 \quad v_0 = v_2 \quad v_1 = v_2 \quad v_1 = v_3 \quad v_2 \neq v_4 \quad v_4 = v_5 \quad v_4 = v_6 \quad v_6 \neq v_7 \quad v_7 \neq v_2$



satisfiable

Outline

Aim

build theory solver for theory of equality with uninterpreted functions (EUF)

Definitions (Terms)

number of arguments

- set of function symbols $\mathcal{F}$ with fixed arity
- set of variables V
- terms $\mathcal{T}(\mathcal{F}, V)$ are built according to grammar

$$t \quad ::= \quad x \mid c \mid f(\underbrace{t, \dots, t}_{n})$$

  if $x \in V$, c is constant, and $f \in \mathsf{F}$ has arity n

- subterms

$$\mathcal{S}ub(t) = \begin{cases} \{t\} & \text{if } t \in V \\ \{t\} \cup \bigcup_i \mathcal{S}ub(t_i) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

Example

- for $\mathcal{F} = \{\mathsf{f}/1, \mathsf{g}/2, \mathsf{a}/0\}$ and $x, y \in V$ have terms a, $\mathsf{f}(x)$, $\mathsf{f}(\mathsf{a})$, $\mathsf{g}(x, \mathsf{f}(y))$, ...
- for $t = \mathsf{g}(\mathsf{g}(x, x), \mathsf{f}(\mathsf{f}(\mathsf{a})))$ have $\mathcal{S}ub(t) = \{t,\ \mathsf{g}(x, x),\ x,\ \mathsf{f}(\mathsf{f}(\mathsf{a})),\ \mathsf{f}(\mathsf{a}),\ \mathsf{a}\}$

Congruence Closure

Input: set of equations E and equation $s = t$ (without variables, only constants)

Output: $s = t$ is *implied* ($E \vDash_{EUF} s = t$) or *not implied* ($E \nvDash_{EUF} s = t$)

1. build congruence classes
   (a) collect all subterms of terms in $E \cup \{s = t\}$
   (b) put different subterms of $E \cup \{s = t\}$ in separate sets
   (c) merge sets $\{\dots, t_1, \dots\}$ and $\{\dots, t_2, \dots\}$ for all $t_1 = t_2$ in E
   (d) merge sets $\{\dots, f(t_1, \dots, t_n), \dots\}$ and $\{\dots, f(u_1, \dots, u_n), \dots\}$
   if t_i and u_i belong to same set for all $1 \leqslant i \leqslant n$
   (e) repeat (d) until no change
1. if s and t belong to same set then return *implied* else return *not implied*

Example (1)

- given set of equations E

$$\mathsf{f(f(f(a))) = g(f(g(f(b))))} \quad \mathsf{f(g(f(b))) = f(a)} \quad \mathsf{g(g(b)) = g(f(a))} \quad \mathsf{g(a) = b}$$

  and test equation $\mathsf{f(a) = g(a)}$

- sets

  1. $\{\, \mathsf{a} \,\}$
  2. $\{\, \mathsf{f(a)}, \mathsf{f(g(f(b)))} \,\}$
  3. $\{\, \mathsf{b}, \mathsf{g(a)} \,\}$
  4. $\{\, \mathsf{g(b)} \,\}$
  5. $\{\, \mathsf{f(f(a))} \,\}$
  6. $\{\, \mathsf{f(f(f(a)))}, \mathsf{g(f(g(f(b))))}, \mathsf{g(g(b))}, \mathsf{g(f(a))} \,\}$
  7. $\{\, \mathsf{f(b)} \,\}$
  8. $\{\, \mathsf{g(f(b))} \,\}$

- conclusion: $E \nvDash_{EUF} \mathsf{f(a) = g(a)}$

Example (2)

- given set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) = \mathsf{a}$$

  and test equaton $\mathsf{f}(\mathsf{a}) = \mathsf{a}$

- $\{\, \mathsf{a}, \mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{f}(\mathsf{a})), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a})))), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) \,\}$

- conclusion: $E \vDash_{EUF} \mathsf{f}(\mathsf{a}) = \mathsf{a}$

Ok, But How About a Solver for EUF?

Assume conjunction of EUF literals φ with free variables $x_1, \dots, x_n$.

Definition (Skolemization)

$\widehat{\varphi} = \varphi[x_1 \mapsto c_1, \dots, x_n \mapsto c_n]$ where $c_1, \dots, c_n$ are distinct fresh constants

Lemma

φ is EUF-satisfiable iff $\widehat{\varphi}$ is EUF-satisfiable

Assumption

assume that $=$ is the only predicate in φ

Remark

if φ contains *n*-ary predicate P different from equality:

- add new constant *true* and *n*-ary function f_P
- replace $P(t_1, \dots, t_n)$ by $f_P(t_1, \dots, t_n) = \mathit{true}$
- replace $P(t_1, \dots, t_n)$ by $f_P(t_1, \dots, t_n) \neq \mathit{true}$

Assume conjunction of equations and inequalities φ with free variables $x_1, \dots, x_n$.

P is set of equations,
N is set of inequalities

Deciding satisfiability of set of EUF literals

split $\varphi = (\bigwedge P) \wedge (\bigwedge N)$ into positive literals P and negative literals N

$\varphi = (\bigwedge P) \wedge (\bigwedge N)$	EUF-unsatisfiable	
$\iff (\bigwedge \widehat{P}) \wedge (\bigwedge \widehat{N})$	EUF-unsatisfiable	skolemization
$\iff \neg \Big((\bigwedge \widehat{P}) \wedge (\bigwedge \widehat{N}) \Big)$	EUF-valid	φ unsat iff $\neg\varphi$ valid
$\iff \bigwedge \widehat{P} \to \bigvee_{l \in \widehat{N}} \neg l$	EUF-valid	
$\iff \exists\, s \neq t$ in $\widehat{N}$ such that $\bigwedge \widehat{P} \to s = t$ is EUF-valid		semantics of $\vee$
$\iff \exists\, s \neq t$ in $\widehat{N}$ such that $\bigwedge \widehat{P} \vDash_{EUF} s = t$		semantics of $\vDash_{EUF}$

Obtained Satisfiability Check

$$(\bigwedge P) \wedge (\bigwedge N) \text{ unsatisfiable} \iff \exists\, s \neq t \text{ in } \widehat{N} \text{ such that } \bigwedge \widehat{P} \vDash_T s = t$$

Example

1. $\mathsf{g}(\mathsf{a}) = \mathsf{c} \wedge \mathsf{f}(\mathsf{g}(\mathsf{a})) \neq \mathsf{f}(\mathsf{c}) \wedge \mathsf{c} \neq \mathsf{d}$
   - split into $P = \{\mathsf{g}(\mathsf{a}) = \mathsf{c}\}$ and $N = \{\mathsf{f}(\mathsf{g}(\mathsf{a})) \neq \mathsf{f}(\mathsf{c}),\ \mathsf{c} \neq \mathsf{d}\}$
   - have $\mathsf{g}(\mathsf{a}) = \mathsf{c} \vDash_T \mathsf{f}(\mathsf{g}(\mathsf{a})) = \mathsf{f}(\mathsf{c})$, so unsatisfiable

2. $\mathsf{g}(\mathsf{a}) = \mathsf{c} \wedge \mathsf{f}(\mathsf{g}(\mathsf{a})) = \mathsf{f}(\mathsf{c}) \wedge \mathsf{g}(\mathsf{a}) = \mathsf{d} \wedge \mathsf{c} \neq \mathsf{d}$
   - split into $P = \{\mathsf{g}(\mathsf{a}) = \mathsf{c},\ \mathsf{f}(\mathsf{g}(\mathsf{a})) = \mathsf{f}(\mathsf{c}),\ \mathsf{g}(\mathsf{a}) = \mathsf{d}\}$ and $N = \{\mathsf{c} \neq \mathsf{d}\}$
   - have $\mathsf{g}(\mathsf{a}) = \mathsf{c}, \mathsf{f}(\mathsf{g}(\mathsf{a})) = \mathsf{f}(\mathsf{c}), \mathsf{g}(\mathsf{a}) = \mathsf{d} \vDash_T \mathsf{c} = \mathsf{d}$, so unsatisfiable

3. $\mathsf{g}(\mathsf{a}) = \mathsf{c} \wedge \mathsf{c} = \mathsf{d} \wedge \mathsf{f}(x) = x \wedge \mathsf{d} \neq \mathsf{g}(x) \wedge \mathsf{f}(x) \neq \mathsf{d}$
   - $P = \{\mathsf{g}(\mathsf{a}) = \mathsf{c},\ \mathsf{c} = \mathsf{d},\ \mathsf{f}(x) = x\}$ and $N = \{\mathsf{d} \neq \mathsf{g}(x),\ \mathsf{f}(x) \neq \mathsf{d}\}$
   - skolemize $P = \{\mathsf{g}(\mathsf{a}) = \mathsf{c},\ \mathsf{c} = \mathsf{d},\ \mathsf{f}(\mathsf{e}) = \mathsf{e}\}$, $N = \{\mathsf{d} \neq \mathsf{g}(\mathsf{e}),\ \mathsf{f}(\mathsf{e}) \neq \mathsf{d}\}$
     - $\mathsf{g}(\mathsf{a}) = \mathsf{c},\ \mathsf{c} = \mathsf{d},\ \mathsf{f}(\mathsf{e}) = \mathsf{e} \nvDash_T \mathsf{d} = \mathsf{g}(\mathsf{e})$
     - $\mathsf{g}(\mathsf{a}) = \mathsf{c},\ \mathsf{c} = \mathsf{d},\ \mathsf{f}(\mathsf{e}) = \mathsf{e} \nvDash_T \mathsf{f}(\mathsf{e}) = \mathsf{d}$

   so satisfiable

Outline

- Summary of Last Week
- Deciding EQ: Equality Graphs
- Deciding EUF: Congruence Closure
- Correctness of DPLL(T)
- Some More Practical SMT

Definition (Basic DPLL(T))
system $\mathcal{B}$ consists of unit propagate, decide, fail, T-backjump, and T-propagate

Definition (Full DPLL(T))
system $\mathcal{D}$ extends $\mathcal{B}$ by T-learn, T-forget, and restart

Lemma
if $\parallel F \Longrightarrow^*_{\mathcal{D}} M \parallel G$ *then*

- *all atoms in M and G are atoms in F*
- *M does not contain complementary literals, and every literal at most once*
- *G is T-equivalent to F ($F \equiv_T G$)*
- *if $M = M_0\ l_1^d\ M_1\ l_2^d\ M_2 \dots\ l_k^d\ M_k$ with $l_1, \dots, l_k$ all the decision literals then $F, l_1, \dots, l_i \vDash_T M_i$ for all $0 \leqslant i \leqslant k$*

Consider derivation with final state S_n:

$$\parallel F \quad \Longrightarrow_{\mathcal{D}} \quad S_1 \quad \Longrightarrow_{\mathcal{D}} \quad S_2 \quad \Longrightarrow_{\mathcal{D}} \quad \dots \quad \Longrightarrow_{\mathcal{D}} \quad S_n$$

Theorem

if $S_n =$ FailState then F is T-unsatisfiable

Proof.

- must have $\parallel F \Longrightarrow^*_{\mathcal{D}} M \parallel F' \overset{\text{fail}}{\Longrightarrow}_{\mathcal{D}}$ FailState, so $M \vDash \neg C$ for some C in F'
- M cannot contain decision literals (otherwise T-backjump applicable)
- by Lemma before, $F' \vDash_T M$, so $F' \vDash_T \neg C$
- also have $F' \vDash_T C$ because C is in F' and $F \equiv_T F'$ so T-inconsistent ■

Theorem

if $S_n = M \parallel F'$ and M is T-consistent then F is T-satisfiable and $M \vDash_T F$

Proof.

- S_n is final, so all literals of F' are defined in M (otherwise decide applicable)
- $\nexists$ clause C in F' such that $M \vDash \neg C$ (otherwise backjump or fail applicable)
- so $M \vDash F'$ and by T-consistency $M \vDash_T F'$
- have $F \equiv_T F'$ so M also T-satisfies F ■

Theorem (Termination)

$$\Gamma: \quad \| F \Longrightarrow_{\mathcal{D}}^{*} S_0 \Longrightarrow_{\mathcal{D}}^{*} S_1 \Longrightarrow_{\mathcal{D}}^{*} \dots$$

is finite if

- *there is no infinite sub-derivation of only* T*-learn and* T*-forget steps, and*
- *for every sub-derivation*

$$S_i \overset{restart}{\Longrightarrow}_{\mathcal{D}} S_{i+1} \Longrightarrow_{\mathcal{D}}^{*} S_j \overset{restart}{\Longrightarrow}_{\mathcal{D}} S_{j+1} \Longrightarrow_{\mathcal{D}}^{*} S_k \overset{restart}{\Longrightarrow}_{\mathcal{D}} S_{k+1}$$

with no restart steps in $S_{i+1} \Longrightarrow_{\mathcal{D}}^{*} S_j$ *and* $S_{j+1} \Longrightarrow_{\mathcal{D}}^{*} S_k$*:*

  - *there are more* $\mathcal{B}$*-steps in* $S_j \Longrightarrow_{\mathcal{D}}^{*} S_k$ *than in* $S_i \Longrightarrow_{\mathcal{D}}^{*} S_j$*, or*
  - *a clause is learned in* $S_j \Longrightarrow_{\mathcal{D}}^{*} S_k$ *that is never forgotten in* Γ

Proof.
similar as for DPLL:

- restart is applied with increasing periodicity, or
- otherwise clause is learned (and there are only finitely many clauses) ■

Integer Arithmetic in python/z3

```
from z3 import *
a = Int('a') # create integer variables
b = Int('b')
c = Int('c')

phi = And(c > 0, b >= 0, a < -1) # some comparisons
psi = (a == If (b == c, b - 2, c - 4)) # if-then-else expression
print(phi)
solver = Solver()
solver.add(phi, psi) # assert constraints
solver.add(a + b < 2 * c) # arithmetic

result = solver.check() # check for satisfiability
if result == z3.sat:
  model = solver.model() # get valuation
  print model[a], model[b], model[c] # -3 0 1
```