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SAT and SMT Solving

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lecture 7
WS 2022

Outline

- Summary of Last Week
- Linear Arithmetic
- Simplex Algorithm

Deciding the Theory of Equality

Definition

- equality logic formula φ_{EQ} is set of equations and inequalities between variables
- write $\mathcal{V}ar(\varphi_{\mathsf{EQ}})$ for set of variables occurring in φ_{EQ}

Definition

equality graph for φ_{EQ} is undirected graph $(V, E_{=}, E_{\neq})$ with two kinds of edges

- nodes $V = \mathcal{V}ar(\varphi_{\mathsf{EQ}})$
- $(x, y) \in E_{=}$ iff $x = y$ in φ_{EQ} — equality edge
- $(x, y) \in E_{\neq}$ iff $x \neq y$ in φ_{EQ} — inequality edge

Definition (Contradictory cycle)

contradictory cycle is simple cycle in equality graph with one $E_{\neq}$ edge
and all others $E_{=}$ edges

Theorem

φ_{EQ} is satisfiable iff its equality graph has no contradictory cycle

Deciding the Theory of Equality with Uninterpreted Functions

Remark

- can assume that $=$ is the only predicate in φ
- can replace variables by constants (Skolemization)

Congruence Closure

Input: set of equations E and equation $s = t$ (without variables, only constants)

Output: $s = t$ is *implied* ($E \models_{EUF} s = t$) or *not implied* ($E \not\models_{EUF} s = t$)

1. build congruence classes
   (a) put different subterms of terms in $E \cup \{s \approx t\}$ in separate sets
   (b) merge sets $\{\ldots, t_1, \ldots\}$ and $\{\ldots, t_2, \ldots\}$ for all $t_1 \approx t_2$ in E
   (c) merge sets $\{\ldots, f(t_1, \ldots, t_n), \ldots\}$ and $\{\ldots, f(u_1, \ldots, u_n), \ldots\}$ if t_i and u_i belong to same set for all $1 \leqslant i \leqslant n$, repeatedly
2. if s and t belong to same set then return *implied* else return *not implied*

Satisfiability Check for EUF

$$(\bigwedge P) \wedge (\bigwedge N) \text{ unsatisfiable} \iff \exists\, s \neq t \text{ in } \widehat{N} \text{ such that } \bigwedge \widehat{P} \models_{EUF} s = t$$

Correctness of DPLL(T)

Definition (DPLL(T) systems)

- basic system $\mathcal{B}$: unit propagate, decide, fail, T-backjump, T-propagate
- full system $\mathcal{F}$: $\mathcal{B}$ plus T-learn, T-forget, and restart

Theorem (Correctness)

For derivation with final state S_n:

$$\| F \quad \Longrightarrow_{\mathcal{F}} \quad S_1 \quad \Longrightarrow_{\mathcal{F}} \quad S_2 \quad \Longrightarrow_{\mathcal{F}} \quad \dots \quad \Longrightarrow_{\mathcal{F}} \quad S_n$$

- *if $S_n =$* FailState *then F is T-unsatisfiable*
- *if $S_n = M \| F'$ and M is T-consistent then F is T-satisfiable and $M \models_T F$*

Theorem (Termination)

Γ: $\| F \Longrightarrow^*_{\mathcal{F}} S_0 \Longrightarrow^*_{\mathcal{F}} S_1 \Longrightarrow^*_{\mathcal{F}} \dots$ *is finite if*

- *there is no infinite sub-derivation of only T-learn and T-forget steps, and*
- *for every sub-derivation $S_i \overset{restart}{\Longrightarrow}_{\mathcal{F}} S_{i+1} \Longrightarrow^*_{\mathcal{F}} S_j \overset{restart}{\Longrightarrow}_{\mathcal{F}} S_{j+1} \Longrightarrow^*_{\mathcal{F}} S_k$*
  - *there are more $\mathcal{B}$-steps in $S_j \Longrightarrow^*_{\mathcal{F}} S_k$ than in $S_i \Longrightarrow^*_{\mathcal{F}} S_j$, or*
  - *a clause is learned in $S_i \Longrightarrow^*_{\mathcal{F}} S_j$ that is never forgotten in Γ*

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- signature
  - binary predicates $<$ and $=$
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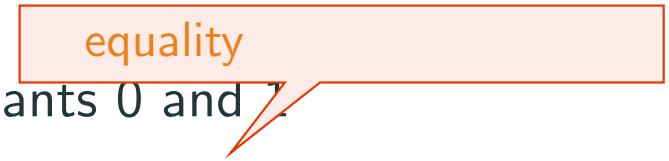

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commutative group with $+$

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$<$ is total order

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non-triviality

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$+$ and $<$ are compatible

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division with remainder

infinitely many axioms for all $n > 0$

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- *$\mathbb{Z}$ with usual interpretations is model of LIA*
- *and it is unique model up to elementary equivalence*

i.e., same formulas hold

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- $x+y+z = 1+1 \ \wedge\ y < z \ \wedge\ -1 < y$ LIA-satisfiable, $v(x) = v(y) = 0$, $v(z) = 2$

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- $x < 1 \wedge 1 < x + x$ LIA-unsatisfiable

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Example (LIA with syntactic sugar)

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- $x < 1 \wedge 2x > 1$
- $7x = 41$

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LIA is decidable and NP-complete

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infinitely many axioms

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- *and it is the single unique model up to elementary equivalence*

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Syntactic Sugar

use same shorthands as for LIA, plus

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LRA is decidable in polynomial time

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1826 Fourier and Motzkin (1936) developed elimination algorithm for LRA

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for linear objective function c, matrix A, vector b, and vector of variables $\overline{x}$

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2000- SMT solvers use DPLL(T) version to solve satisfiability problem

$$A\overline{x} \leqslant b$$

Outline

- Summary of Last Week
- Linear Arithmetic
- Simplex Algorithm

Aim

build theory solver for linear rational arithmetic (LRA):

decide whether set of linear (in)equalities is satisfiable over $\mathbb{Q}$

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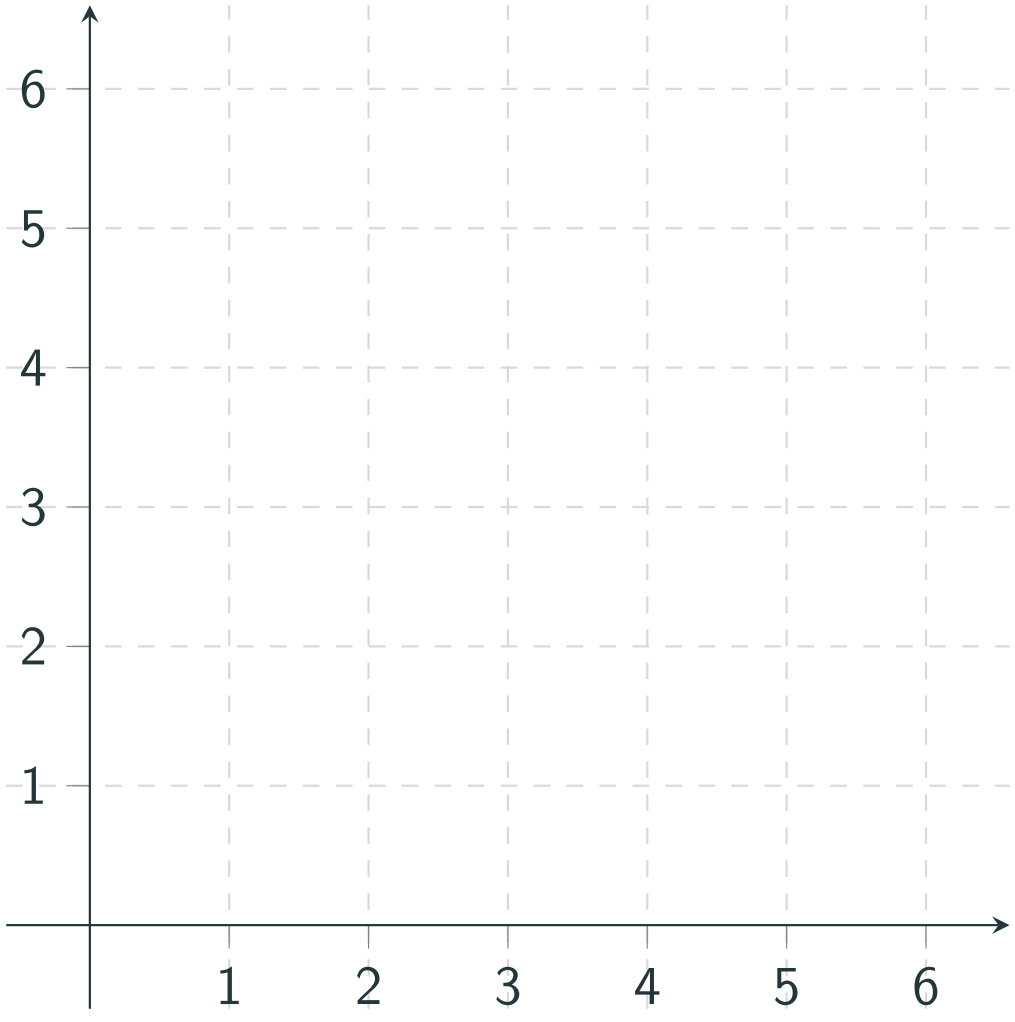
Disclaimer: Effects and Side Effects

- guaranteed to solve all your real arithmetic problems
- consuming Simplex can cause initial dizzyness
- in some cases solving systems of linear inequalities can become addictive

Simplex, Visually

- constraints

$$x - y \geqslant -1$$
$$y \leqslant 4$$
$$x + y \geqslant 6$$
$$3x - y \leqslant 7$$



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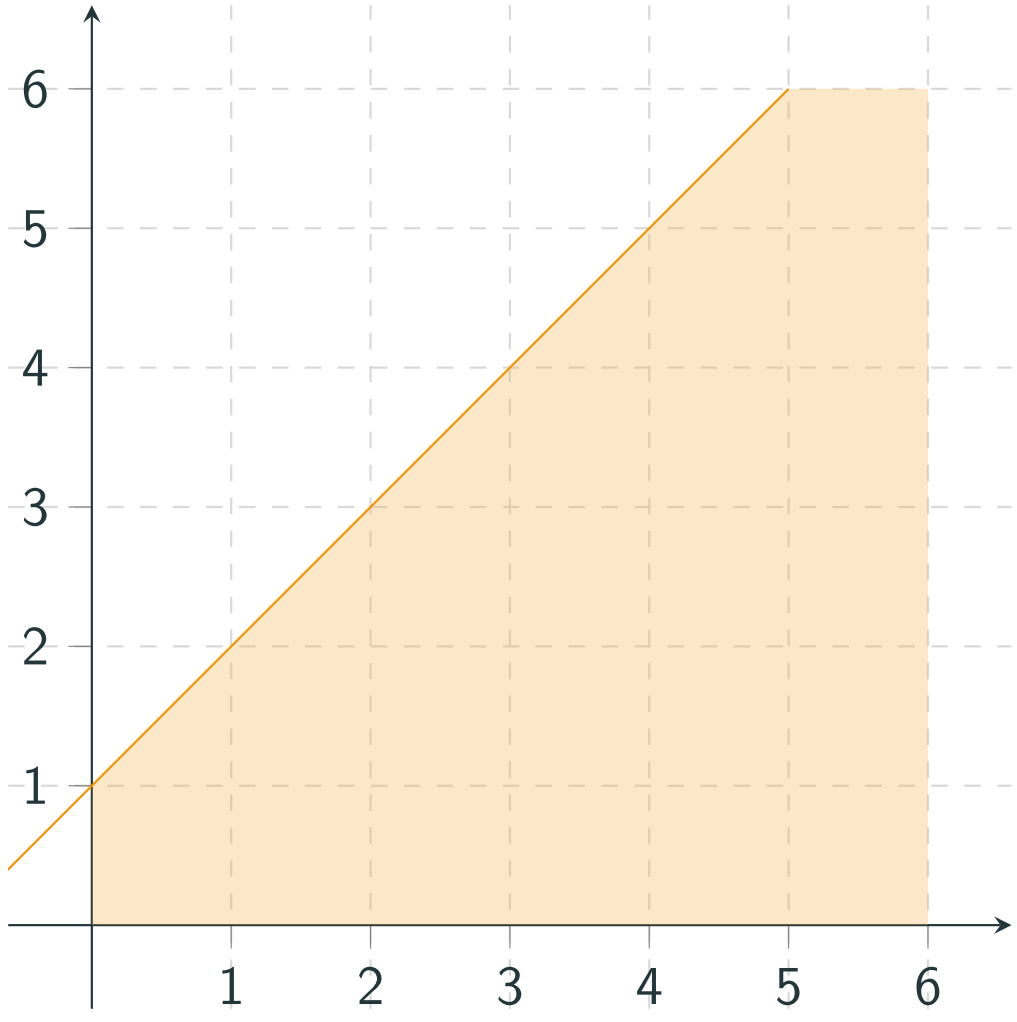

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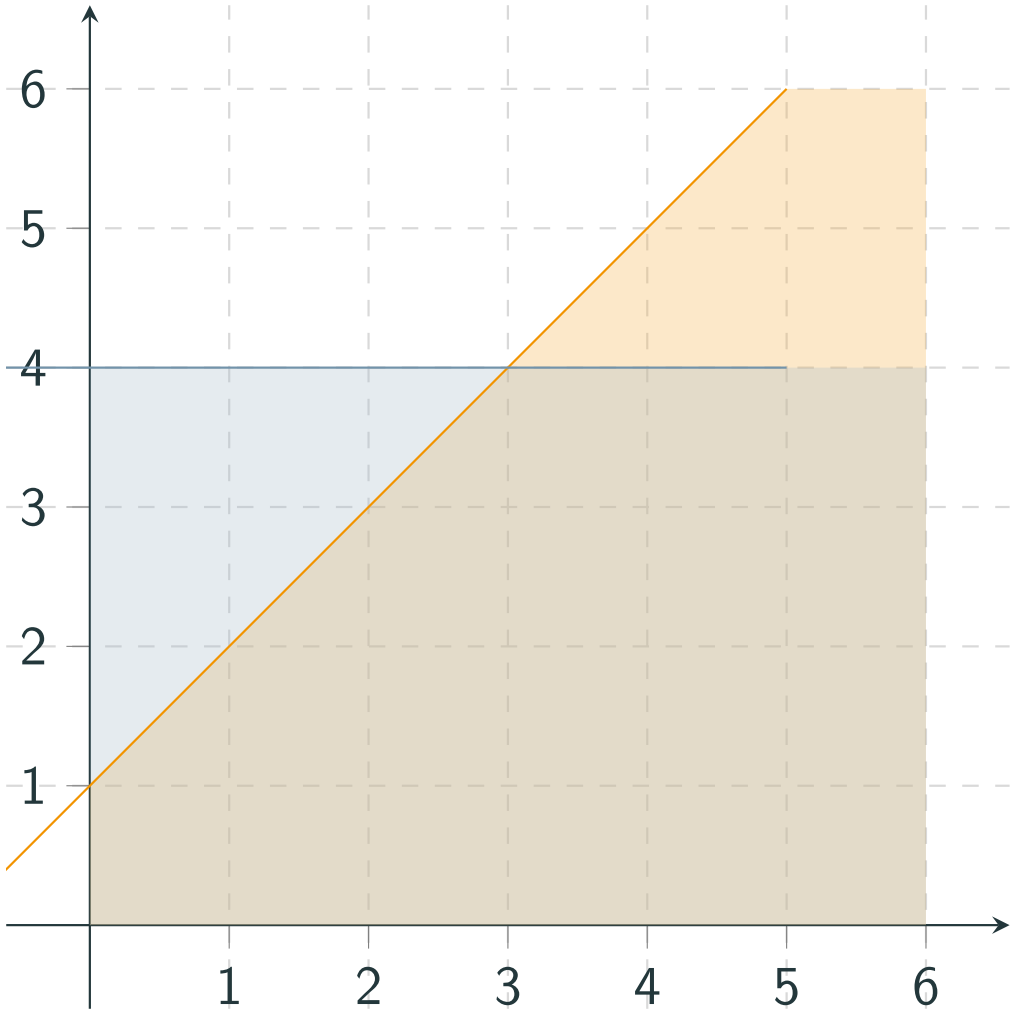

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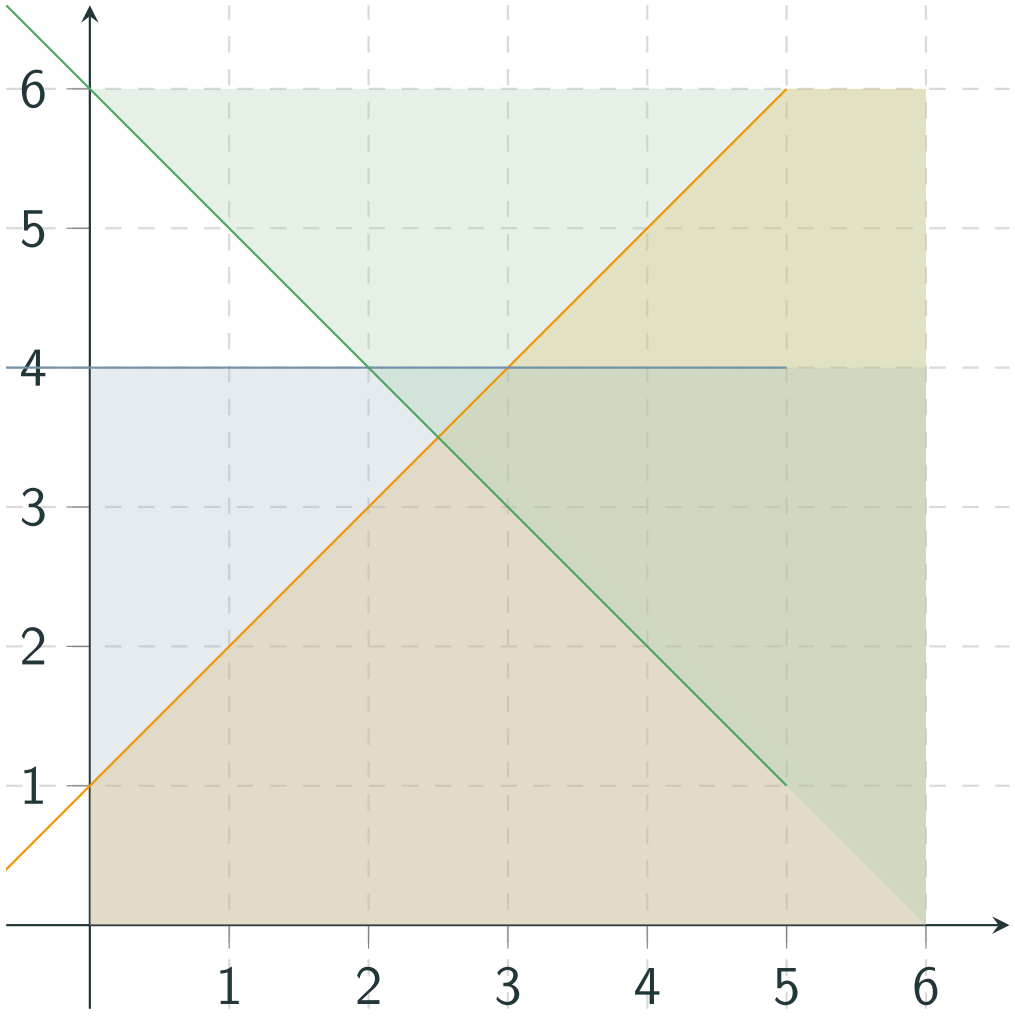

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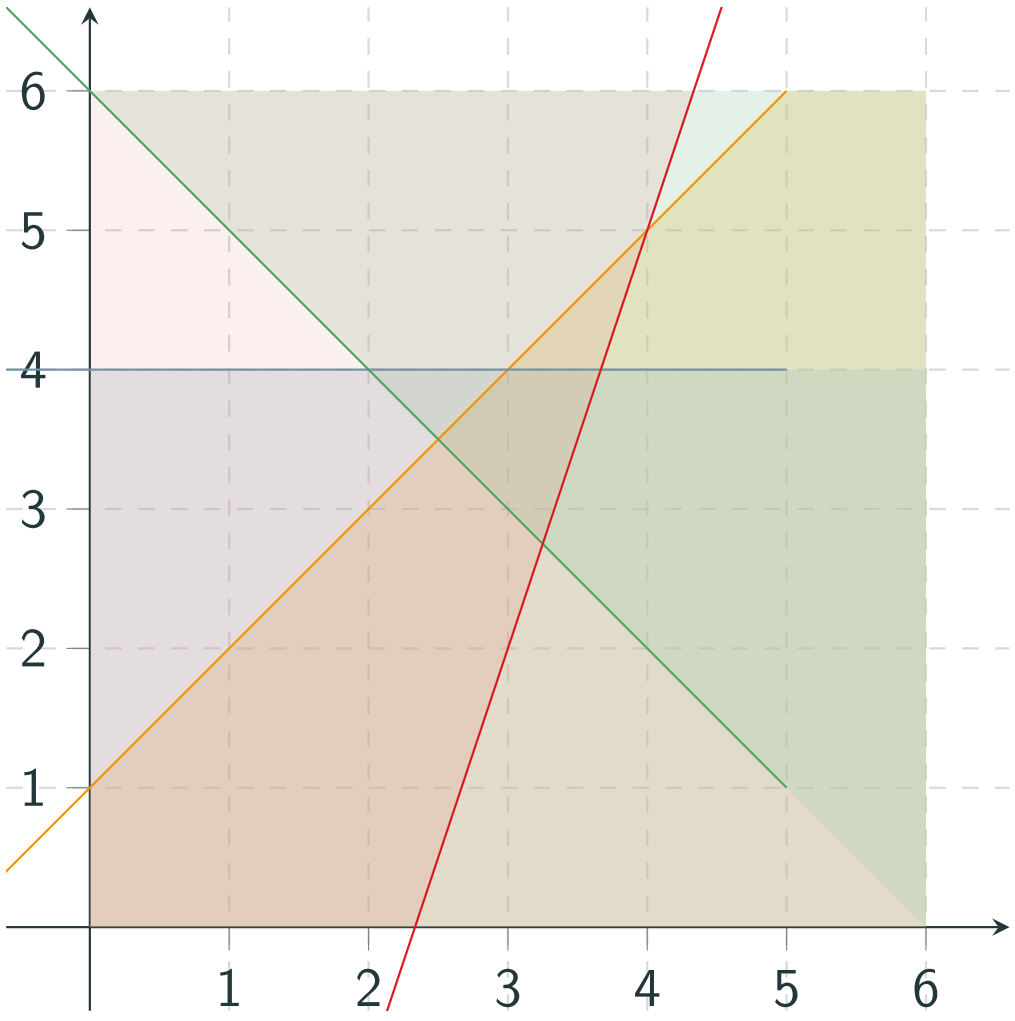

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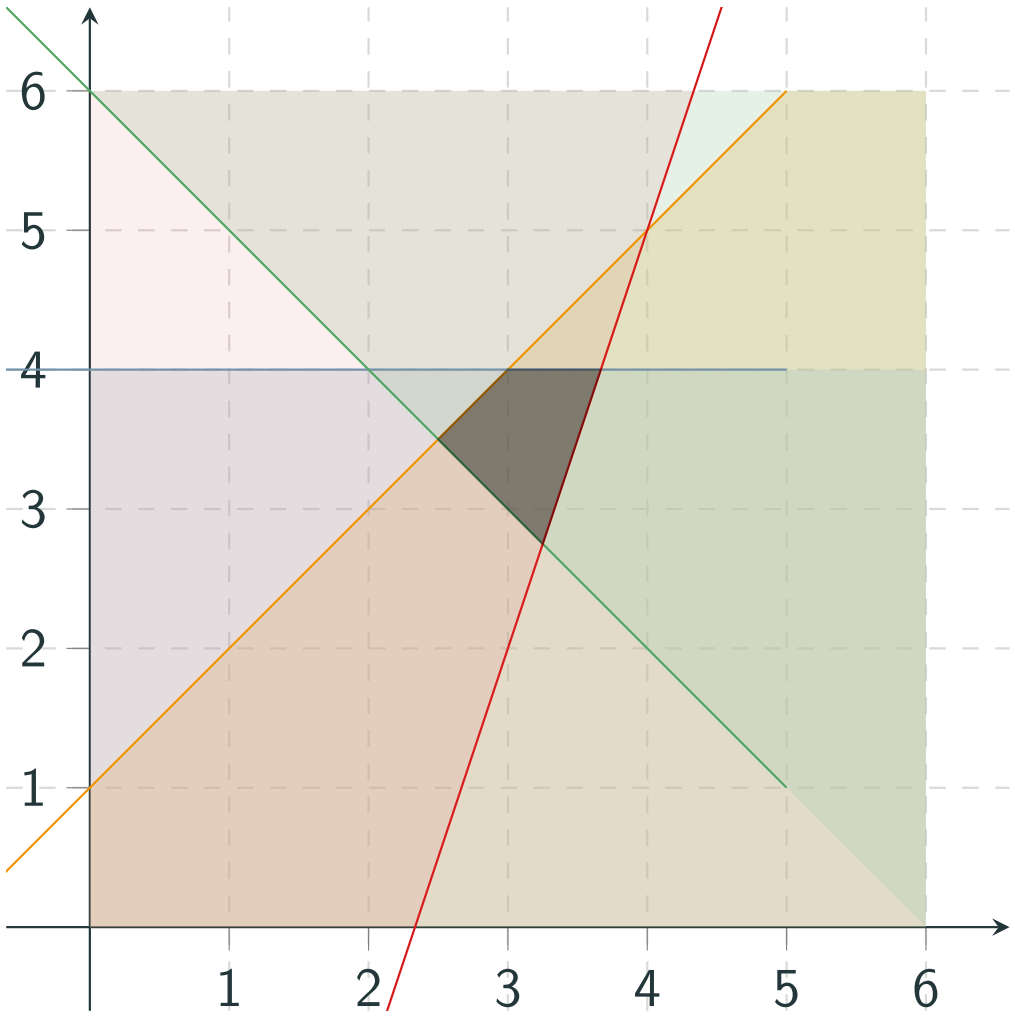

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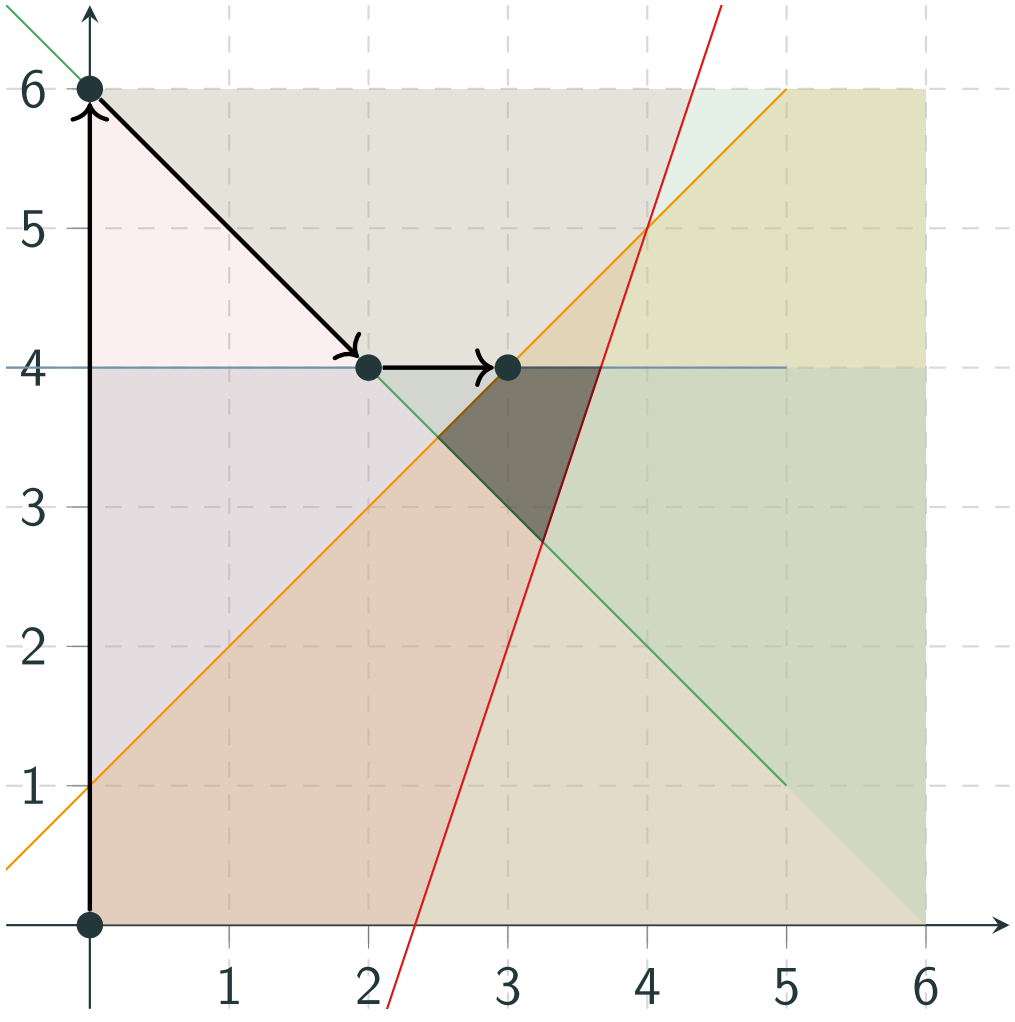
- solution space
- Simplex algorithm: improve assignment in 4 iterations
  - $x = 0,\ y = 0$
  - $x = 0,\ y = 6$
  - $x = 2,\ y = 4$
  - $x = 3,\ y = 4$



Definition (Problem in general form)

- variables $x_1, \ldots, x_n$
- m equalities for $a_{ij} \in \mathbb{Q}$

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- represent equalities by $m \times (n+m)$ matrix A such that $A \cdot \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = 0$

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- simplified matrix presentation

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

Representation

- represent equalities by $m \times (n+m)$ matrix A such that $A \cdot \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = 0$

$$\begin{aligned} -x + y - s_1 &= 0 \quad s_1 \leqslant 1 \\ y - s_2 &= 0 \quad s_2 \leqslant 4 \\ -x - y - s_3 &= 0 \quad s_3 \leqslant -6 \\ 3x - y - s_4 &= 0 \quad s_4 \leqslant 7 \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 3 & -1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{matrix} s_1 \leqslant 1 \\ s_2 \leqslant 4 \\ s_3 \leqslant -6 \\ s_4 \leqslant 7 \end{matrix}$$

- simplified matrix presentation

dependent variables $\rightarrow$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

Representation

- represent equalities by $m \times (n+m)$ matrix A such that $A \cdot \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = 0$

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- simplified matrix presentation

$$\begin{matrix} \text{dependent variables} \rightarrow \\ \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} \quad \begin{matrix} x & y \\ \end{matrix} \quad \leftarrow \text{independent variables}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix}$$

Representation

- represent equalities by $m \times (n+m)$ matrix A such that $A \cdot \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = 0$

$$\begin{aligned} -x + y - s_1 &= 0 \quad s_1 \leqslant 1 \\ y - s_2 &= 0 \quad s_2 \leqslant 4 \\ -x - y - s_3 &= 0 \quad s_3 \leqslant -6 \\ 3x - y - s_4 &= 0 \quad s_4 \leqslant 7 \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 3 & -1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{matrix} s_1 \leqslant 1 \\ s_2 \leqslant 4 \\ s_3 \leqslant -6 \\ s_4 \leqslant 7 \end{matrix}$$

- simplified matrix presentation

$$\begin{matrix} & x & y \\ s_1 & -1 & 1 \\ s_2 & 0 & 1 \\ s_3 & -1 & -1 \\ s_4 & 3 & -1 \end{matrix}$$

dependent variables $\rightarrow$

$\leftarrow$ independent variables

Notation

- simplified matrix is called tableau
- D is set of dependent (or basic) variables, in tableau listed on the left
- I is set of independent (or non-basic) variables, in tableau on top)

DPLL(T) Simplex Algorithm

Input: conjunction of LRA literals φ without $<$, $>$, $\neq$

Output: satisfiable or unsatisfiable

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2. fix order on variables and assign 0 to each variable

DPLL(T) Simplex Algorithm

Input: conjunction of LRA literals φ without $<$, $>$, $\neq$

Output: satisfiable or unsatisfiable

1. transform φ into general form and construct tableau
2. fix order on variables and assign 0 to each variable
3. if all dependent variables satisfy their bounds then return satisfiable

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   (i.e., look for y whose value can be changed such that x is within b)

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   (i.e., look for y whose value can be changed such that x is within b)
6. return unsatisfiable if no such variable exists

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5. search for suitable variable $y \in I$ for pivoting with x
   (i.e., look for y whose value can be changed such that x is within b)
6. return unsatisfiable if no such variable exists
7. perform pivot operation on x and y
   (i.e., make x independent and y dependent)

DPLL(T) Simplex Algorithm

Input: conjunction of LRA literals φ without $<$, $>$, $\neq$

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6. return unsatisfiable if no such variable exists
7. perform pivot operation on x and y
   (i.e., make x independent and y dependent)
9. improve assignment: set x to b, and update accordingly

DPLL(T) Simplex Algorithm

Input: conjunction of LRA literals φ without $<$, $>$, $\neq$
Output: satisfiable or unsatisfiable

1. transform φ into general form and construct tableau
2. fix order on variables and assign 0 to each variable
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10. go to step 3

DPLL(T) Simplex Algorithm

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7. perform pivot operation on x and y
   (i.e., make x independent and y dependent)
9. improve assignment: set x to b, and update accordingly
10. go to step 3

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	0	0	0	0	0

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	0	0	0	0	0

1. Iteration 1
   - s_3 violates its bounds

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	0	0	0	0	0

1. Iteration 1
   - s_3 violates its bounds
   - decreasing s_3 requires to increase x or y because $s_3 = -x - y$: both suitable since they have no upper bound

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	0	0	0	0	0

1. Iteration 1
   - s_3 violates its bounds
   - decreasing s_3 requires to increase x or y because $s_3 = -x - y$: both suitable since they have no upper bound
   - pivot s_3 with y:

$$\begin{aligned} y &= -x - s_3 & \qquad s_1 &= -2x - s_3 \\ s_2 &= -x - s_3 & \qquad s_4 &= 4x + s_3 \end{aligned}$$

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -1 & -1 \\ 4 & 1 \end{pmatrix} \end{array}$$

bounds

$$s_1 \leqslant 1$$
$$s_2 \leqslant 4$$
$$s_3 \leqslant -6$$
$$s_4 \leqslant 7$$

assignment

x	y	s_1	s_2	s_3	s_4
0	0	0	0	0	0

1. Iteration 1
   - s_3 violates its bounds
   - decreasing s_3 requires to increase x or y because $s_3 = -x - y$: both suitable since they have no upper bound
   - pivot s_3 with y:

$$y = -x - s_3 \qquad s_1 = -2x - s_3$$
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Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -1 & -1 \\ 4 & 1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	0	0	0	0	0

1. Iteration 1
   - s_3 violates its bounds
   - decreasing s_3 requires to increase x or y because $s_3 = -x - y$: both suitable since they have no upper bound
   - pivot s_3 with y:

$$\begin{aligned} y &= -x - s_3 & \qquad s_1 &= -2x - s_3 \\ s_2 &= -x - s_3 & \qquad s_4 &= 4x + s_3 \end{aligned}$$

   - update assignment: set s_3 to violated bound -6 and propagate

$$s_3 = -6$$

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -1 & -1 \\ 4 & 1 \end{pmatrix} \end{array}$$

bounds

$$s_1 \leqslant 1$$
$$s_2 \leqslant 4$$
$$s_3 \leqslant -6$$
$$s_4 \leqslant 7$$

assignment

x	y	s_1	s_2	s_3	s_4
0	6	6	6	-6	-6

1. Iteration 1
   - s_3 violates its bounds
   - decreasing s_3 requires to increase x or y because $s_3 = -x - y$: both suitable since they have no upper bound
   - pivot s_3 with y:

$$y = -x - s_3 \qquad s_1 = -2x - s_3$$
$$s_2 = -x - s_3 \qquad s_4 = 4x + s_3$$

   - update assignment: set s_3 to violated bound -6 and propagate

$$s_3 = -6 \qquad y = 6$$
$$s_1 = 6 \qquad s_2 = 6 \qquad s_4 = -6$$

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -1 & -1 \\ 4 & 1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	6	6	6	-6	-6

2 Iteration 2

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -1 & -1 \\ 4 & 1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	6	6	6	-6	-6

2. Iteration 2
   - s_2 violates its bounds

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -1 & -1 \\ 4 & 1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	6	6	6	-6	-6

2 Iteration 2

- s_2 violates its bounds
- decreasing s_2 requires to increase x or s_3 because $s_2 = -x - s_3$:
  x suitable since unbounded, but s_3 not suitable as already at bound!

Example

tableau

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & s_3 \end{array} \\ \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -1 & -1 \\ 4 & 1 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	6	6	6	-6	-6

2 Iteration 2

- s_2 violates its bounds
- decreasing s_2 requires to increase x or s_3 because $s_2 = -x - s_3$:
  x suitable since unbounded, but s_3 not suitable as already at bound!
- pivot s_2 with x:

$$\begin{aligned} x &= -s_2 - s_3 & s_1 &= -2x - s_3 = 2s_2 + s_3 \\ y &= -x - s_3 = s_2 & s_4 &= 4x + s_3 = -4s_2 - 3s_3 \end{aligned}$$

Example

tableau

$$\begin{array}{c} \\ s_1 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_3 \end{array} \\ \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 1 & 0 \\ -4 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	6	6	6	-6	-6

2 Iteration 2

- s_2 violates its bounds
- decreasing s_2 requires to increase x or s_3 because $s_2 = -x - s_3$:
  x suitable since unbounded, but s_3 not suitable as already at bound!
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Example

tableau

$$\begin{array}{c} \\ s_1 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_3 \end{array} \\ \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 1 & 0 \\ -4 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
0	6	6	6	-6	-6

2 Iteration 2

- s_2 violates its bounds
- decreasing s_2 requires to increase x or s_3 because $s_2 = -x - s_3$:
  x suitable since unbounded, but s_3 not suitable as already at bound!
- pivot s_2 with x:

$$\begin{aligned} x &= -s_2 - s_3 & \qquad s_1 &= -2x - s_3 = 2s_2 + s_3 \\ y &= -x - s_3 = s_2 & \qquad s_4 &= 4x + s_3 = -4s_2 - 3s_3 \end{aligned}$$

- update assignment (to violated bound of s_2)

$$s_2 = 4$$

Example

tableau

$$\begin{array}{c} \\ s_1 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_3 \end{array} \\ \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 1 & 0 \\ -4 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
2	4	2	4	-6	2

2 Iteration 2

- s_2 violates its bounds
- decreasing s_2 requires to increase x or s_3 because $s_2 = -x - s_3$:
  x suitable since unbounded, but s_3 not suitable as already at bound!
- pivot s_2 with x:

$$\begin{aligned} x &= -s_2 - s_3 & s_1 &= -2x - s_3 = 2s_2 + s_3 \\ y &= -x - s_3 = s_2 & s_4 &= 4x + s_3 = -4s_2 - 3s_3 \end{aligned}$$

- update assignment (to violated bound of s_2)

$$\begin{aligned} s_2 &= 4 & x &= 2 \\ s_1 &= 2 & y &= 4 & s_4 &= 2 \end{aligned}$$

Example

tableau

$$\begin{array}{c} \\ s_1 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_3 \end{array} \\ \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 1 & 0 \\ -4 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
2	4	2	4	-6	2

3. Iteration 3
   - s_1 violates its bounds

Example

tableau

$$\begin{array}{c} \\ s_1 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_3 \end{array} \\ \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 1 & 0 \\ -4 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
2	4	2	4	-6	2

3 Iteration 3

- s_1 violates its bounds
- decreasing s_1 requires to decrease s_2 or s_3 because $s_1 = 2s_2 + s_3$: both suitable since they have no lower bound

Example

tableau

	s_2	s_3
s_1	2	1
x	-1	-1
y	1	0
s_4	-4	-3

bounds

$$s_1 \leqslant 1$$
$$s_2 \leqslant 4$$
$$s_3 \leqslant -6$$
$$s_4 \leqslant 7$$

assignment

x	y	s_1	s_2	s_3	s_4
2	4	2	4	-6	2

3 Iteration 3

- s_1 violates its bounds
- decreasing s_1 requires to decrease s_2 or s_3 because $s_1 = 2s_2 + s_3$: both suitable since they have no lower bound
- pivot s_1 with s_3:

$$s_3 = -2s_2 + s_1 \qquad x = s_2 - s_1$$
$$y = s_2 \qquad s_4 = 2s_2 - 3s_1$$

Example

tableau

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
2	4	2	4	-6	2

3 Iteration 3

- s_1 violates its bounds
- decreasing s_1 requires to decrease s_2 or s_3 because $s_1 = 2s_2 + s_3$: both suitable since they have no lower bound
- pivot s_1 with s_3:

$$\begin{aligned} s_3 &= -2s_2 + s_1 & x &= s_2 - s_1 \\ y &= s_2 & s_4 &= 2s_2 - 3s_1 \end{aligned}$$

Example

tableau

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
2	4	2	4	-6	2

3 Iteration 3

- s_1 violates its bounds
- decreasing s_1 requires to decrease s_2 or s_3 because $s_1 = 2s_2 + s_3$: both suitable since they have no lower bound
- pivot s_1 with s_3:

$$s_3 = -2s_2 + s_1 \qquad x = s_2 - s_1$$
$$y = s_2 \qquad s_4 = 2s_2 - 3s_1$$

- update assignment (to violated bound of s_1)

$$s_1 = 1$$

Example

tableau

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} \end{array}$$

bounds

$$\begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

assignment

x	y	s_1	s_2	s_3	s_4
3	4	1	4	-7	5

3 Iteration 3

- s_1 violates its bounds
- decreasing s_1 requires to decrease s_2 or s_3 because $s_1 = 2s_2 + s_3$: both suitable since they have no lower bound
- pivot s_1 with s_3:

$$s_3 = -2s_2 + s_1 \qquad x = s_2 - s_1$$
$$y = s_2 \qquad s_4 = 2s_2 - 3s_1$$

- update assignment (to violated bound of s_1)

$$s_1 = 1 \qquad s_3 = -7$$
$$x = 3 \qquad y = 4 \qquad s_4 = 5$$

Example

tableau

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} \end{array}$$

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3	4	1	4	-7	5

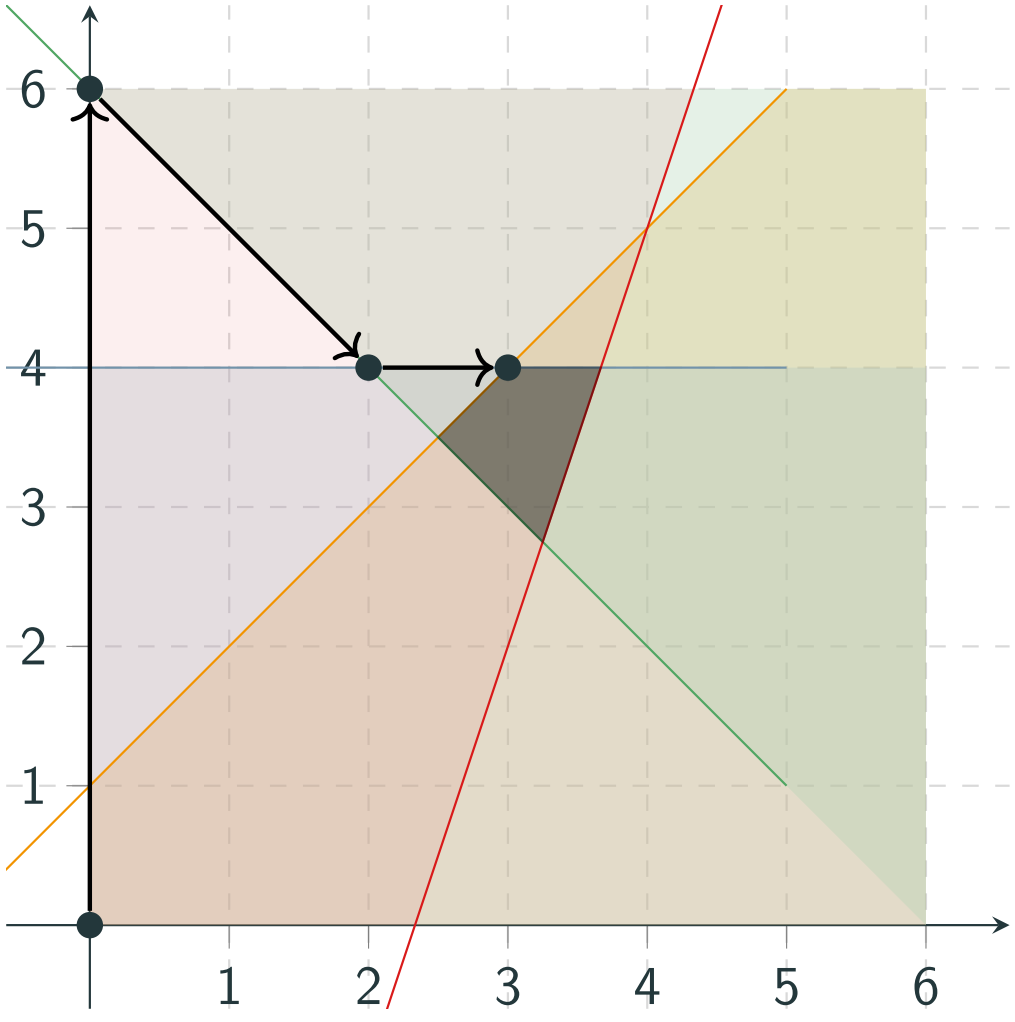
4. Iteration 4
   - all variables satisfy their bounds: satisfiable!

Simplex, Visually

- constraints

$$x - y \geqslant -1$$
$$y \leqslant 4$$
$$x + y \geqslant 6$$
$$3x - y \leqslant 7$$

- solution space
- Simplex algorithm: improve assignment in 4 iterations
  - $x = 0,\ y = 0$
  - $x = 0,\ y = 6$
  - $x = 2,\ y = 4$
  - $x = 3,\ y = 4$



DPLL(T) Simplex Algorithm

$$A\overline{x}_I = \overline{x}_D \qquad (1)$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \qquad (2)$$

Invariant

- (1) is satisfied and (2) holds for all independent variables

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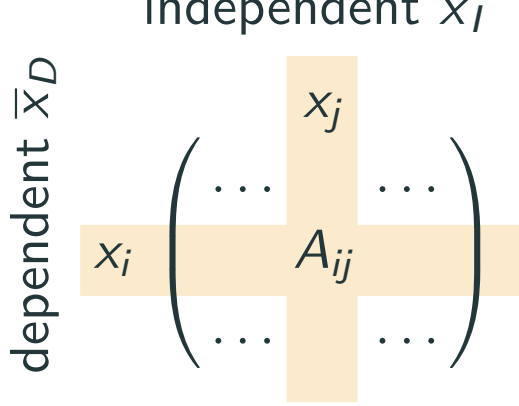

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- (1) is satisfied and (2) holds for all independent variables

Pivoting

- swap dependent x_i and independent x_j, so $x_i \in D$ and $x_j \in I$

$(\star)$

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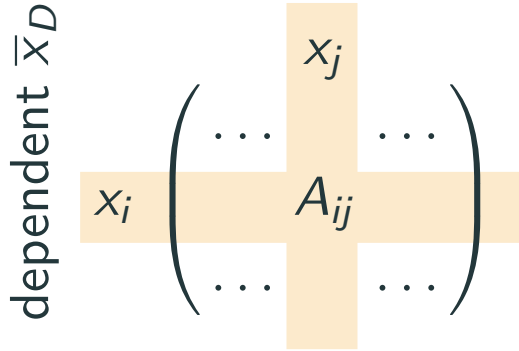

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$$x_i = \sum_{x_k \in I} A_{ik} x_k \qquad (\star)$$

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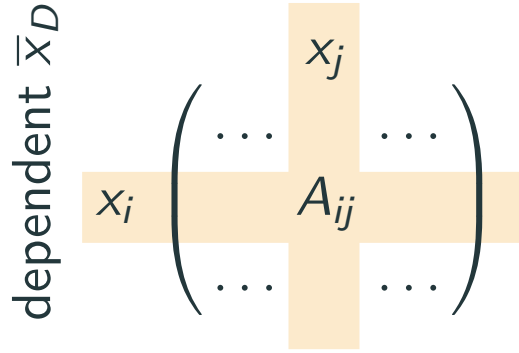

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$$x_i = \sum_{x_k \in I} A_{ik} x_k \qquad \Longrightarrow \qquad x_j = \underbrace{\frac{1}{A_{ij}}\left(x_i - \sum_{x_k \in I - \{x_j\}} A_{ik} x_k\right)}_{t} \qquad (\star)$$

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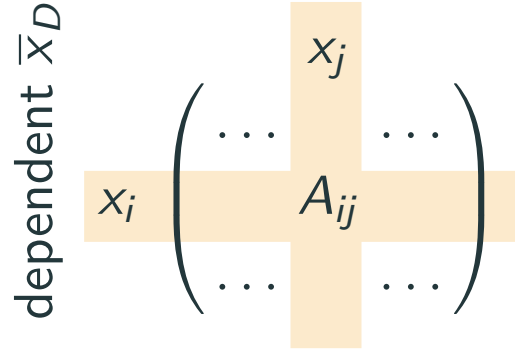

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- new tableau A' consists of $(\star)$ and $x_m = A_{mj}\, t + \sum_{x_k \in I - \{x_j\}} A_{mk} x_k \quad \forall x_m \in D - \{x_i\}$

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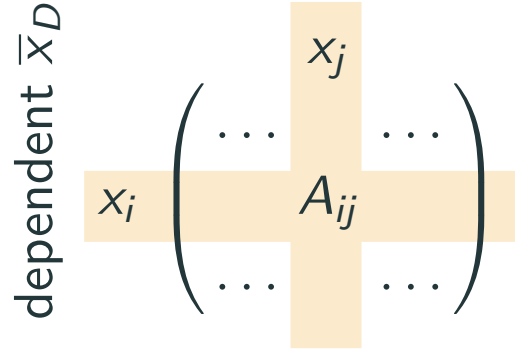

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new row

updated other rows

- new tableau A' consists of $(\star)$ and $x_m = A_{mj}\, t + \sum_{x_k \in I - \{x_j\}} A_{mk} x_k \quad \forall x_m \in D - \{x_i\}$

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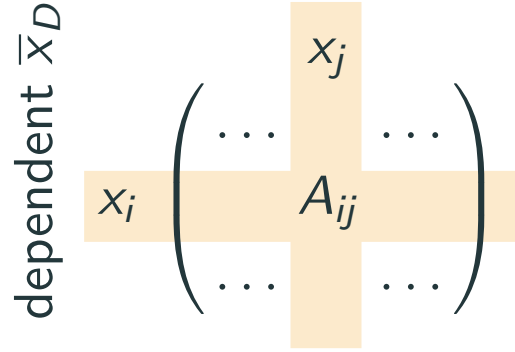

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Update

- assignment of x_i is updated to previously violated bound l_i or u_i,

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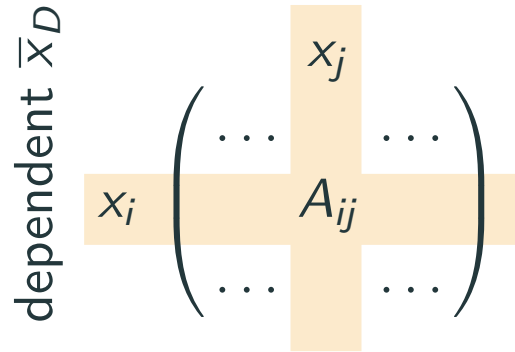

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- new tableau A' consists of $(\star)$ and $x_m = A_{mj}\, t + \sum_{x_k \in I - \{x_j\}} A_{mk} x_k \quad \forall x_m \in D - \{x_i\}$

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of x_k is updated using A' for all $\forall x_m \in D - \{x_i\}$

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- then x_j is suitable for pivoting with x_i if
  - if $x_i < l_i$:

want to increase x_i

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Observation

selecting variables and pivots in unfortunate order may lead to non-termination

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Bland's rule

select variable x_i in step 4 and x_j in step 5 such that (x_i, x_j) is minimal with respect to lexicographic extension of order on variables

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Lemma

- *Simplex terminates if pivot variables are selected according to Bland's rule*
- *problem is satisfiable iff Simplex returns satisfiable*

How to Deal With Strict Inequalities?

replace in LRA formula φ every strict inequality

$$a_1 x_1 + \cdots + a_n x_n < b$$

by non-strict inequality

$$a_1 x_1 + \cdots + a_n x_n \leqslant b - \delta$$

to obtain formula φ_δ in LRA without $<$, and treat δ as variable during Simplex

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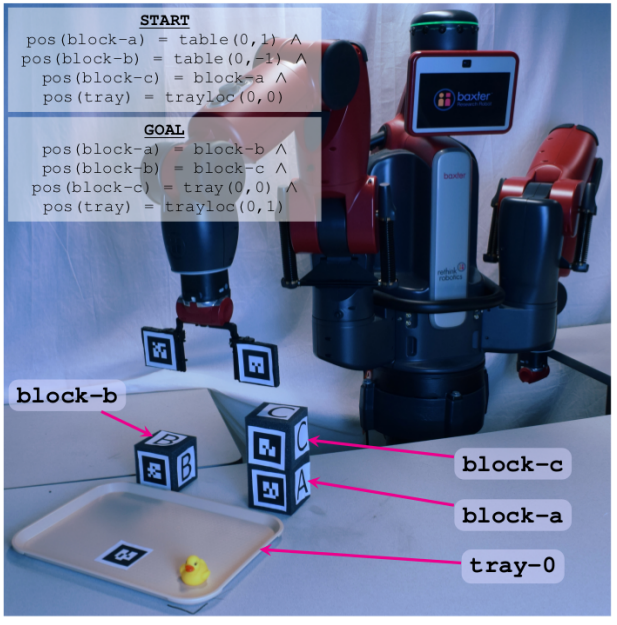
to obtain formula φ_δ in LRA without $<$, and treat δ as variable during Simplex

Lemma

φ *is satisfiable* $\iff$ $\exists$ *rational number* $\delta > 0$ *such that* φ_δ *is satisfiable*

Application: Motion Planning for Robots

- robots needs to plan motions
  to place objects correctly
- instance of *constraint based planning*



Application: Motion Planning for Robots

- robots needs to plan motions
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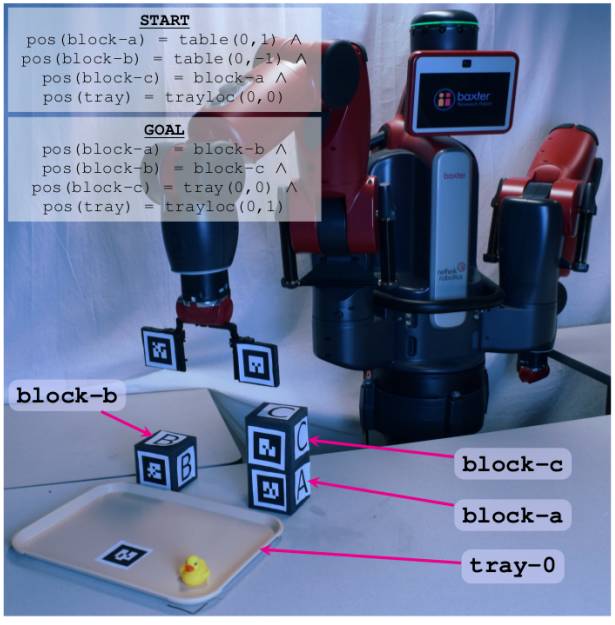
- encoding
  - fix number of time slots $t_1, \ldots, t_n$
  - action variable a_i for time t_i encodes
    which action performed at time t_i
    (one action per time)
  - actions require precondition and imply
    postcondition
  - use arithmetic to minimize path



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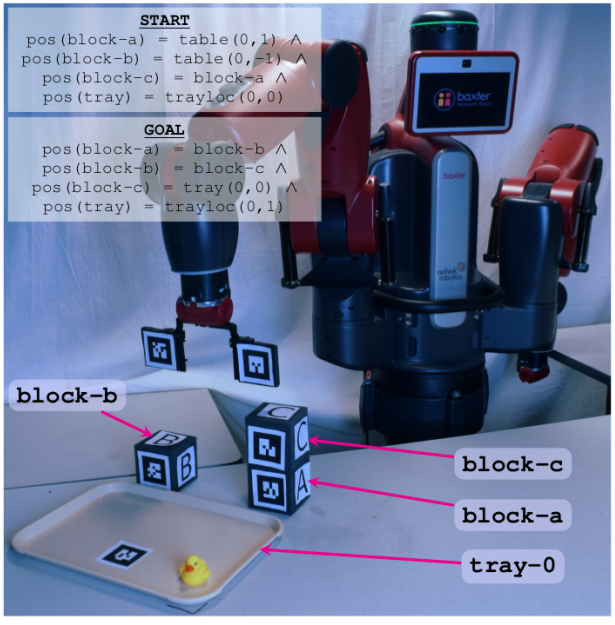

Neil T. Dantam, Zachary K. Kingston, Swarat Chaudhuri, and Lydia E. Kavraki.
Incremental Task and Motion Planning: A Constraint-Based Approach.
In: The International Journal of Robotics Research, 2018.

(Almost) Everything is Better With Arithmetic

LRA and LIA admit more efficient encodings of

- n-queens
- Sudoku
- graph coloring
- Minesweeper
- travelling salesperson
- rabbit problem
- planning problems
- scheduling problems
- component configuration problems
- everything with cardinality constraints
- ...

Bibliography

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In Proc. of International Conference on Computer Aided Verification, pp. 81–94, 2006.

Bruno Dutertre and Leonardo de Moura
Integrating Simplex with DPLL(T)
Technical Report SRI–CSL–06–01, SRI International, 2006

Test on December 2

- 50 minutes
- open (paper) book: bring arbitrary amount of printed paper,
  but use no electronic devices
- questions are like homework exercises:
  e.g., DPLL, implication graphs, give minimal unsatisfiable core of formula, equality graphs, congruence closure, DPLL(T), ... (no Simplex)