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SAT and SMT Solving

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lecture 7
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Outline

- Summary of Last Week
- Linear Arithmetic
- Simplex Algorithm

Deciding the Theory of Equality

Definition

- equality logic formula φ_{EQ} is set of equations and inequalities between variables
- write $\mathcal{V}ar(\varphi_{\mathsf{EQ}})$ for set of variables occurring in φ_{EQ}

Definition

equality graph for φ_{EQ} is undirected graph $(V, E_=, E_\neq)$ with two kinds of edges

- nodes $V = \mathcal{V}ar(\varphi_{\mathsf{EQ}})$
- $(x, y) \in E_=$ iff $x = y$ in φ_{EQ} — equality edge
- $(x, y) \in E_\neq$ iff $x \neq y$ in φ_{EQ} — inequality edge

Definition (Contradictory cycle)

contradictory cycle is simple cycle in equality graph with one $E_\neq$ edge
and all others $E_=$ edges

Theorem

φ_{EQ} is satisfiable iff its equality graph has no contradictory cycle

Deciding the Theory of Equality with Uninterpreted Functions

Remark

- can assume that $=$ is the only predicate in φ
- can replace variables by constants (Skolemization)

Congruence Closure

Input: set of equations E and equation $s = t$ (without variables, only constants)

Output: $s = t$ is *implied* ($E \vDash_{EUF} s = t$) or *not implied* ($E \nvDash_{EUF} s = t$)

1. build congruence classes
   (a) put different subterms of terms in $E \cup \{s \approx t\}$ in separate sets
   (b) merge sets $\{\ldots, t_1, \ldots\}$ and $\{\ldots, t_2, \ldots\}$ for all $t_1 \approx t_2$ in E
   (c) merge sets $\{\ldots, f(t_1, \ldots, t_n), \ldots\}$ and $\{\ldots, f(u_1, \ldots, u_n), \ldots\}$
   if t_i and u_i belong to same set for all $1 \leqslant i \leqslant n$, repeatedly
2. if s and t belong to same set then return *implied* else return *not implied*

Satisfiability Check for EUF

$$(\bigwedge P) \wedge (\bigwedge N) \text{ unsatisfiable} \iff \exists\, s \neq t \text{ in } \widehat{N} \text{ such that } \bigwedge \widehat{P} \vDash_{EUF} s = t$$

Correctness of DPLL(T)

Definition (DPLL(T) systems)

- basic system $\mathcal{B}$: unit propagate, decide, fail, T-backjump, T-propagate
- full system $\mathcal{F}$: $\mathcal{B}$ plus T-learn, T-forget, and restart

Theorem (Correctness)

For derivation with final state S_n:

$$\| \; F \quad \Longrightarrow_{\mathcal{F}} \quad S_1 \quad \Longrightarrow_{\mathcal{F}} \quad S_2 \quad \Longrightarrow_{\mathcal{F}} \quad \dots \quad \Longrightarrow_{\mathcal{F}} \quad S_n$$

- *if* $S_n = \text{FailState}$ *then* F *is* T*-unsatisfiable*
- *if* $S_n = M \| F'$ *and* M *is* T*-consistent then* F *is* T*-satisfiable and* $M \vDash_T F$

Theorem (Termination)

Γ: $\quad \| \; F \Longrightarrow^*_{\mathcal{F}} S_0 \Longrightarrow^*_{\mathcal{F}} S_1 \Longrightarrow^*_{\mathcal{F}} \dots$ *is finite if*

- *there is no infinite sub-derivation of only* T*-learn and* T*-forget steps, and*
- *for every sub-derivation* $S_i \overset{restart}{\Longrightarrow}_{\mathcal{F}} S_{i+1} \Longrightarrow^*_{\mathcal{F}} S_j \overset{restart}{\Longrightarrow}_{\mathcal{F}} S_{j+1} \Longrightarrow^*_{\mathcal{F}} S_k$
  - *there are more* $\mathcal{B}$*-steps in* $S_j \Longrightarrow^*_{\mathcal{F}} S_k$ *than in* $S_i \Longrightarrow^*_{\mathcal{F}} S_j$*, or*
  - *a clause is learned in* $S_j \Longrightarrow^*_{\mathcal{F}} S_k$ *that is never forgotten in* Γ

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Definition (Theory of Linear Arithmetic over $\mathbb{Z}$ (LIA))

- signature
  - binary predicates $<$ and $=$
  - binary function $+$, unary function $-$, constants 0 and 1
- axioms

$$\forall x.\ (x = x) \qquad \forall x\, y.\ (x = y \rightarrow y = x) \qquad \forall x\, y\, z.\ (x = y \wedge y = z \rightarrow x = z)$$

$$\forall x.\ (x + 0 = x) \qquad \forall x\, y.\ (x + y = y + x) \qquad \forall x\, y\, z.\ (x + (y + z) = (x + y) + z)$$

$$\forall x.\ \neg(x < x) \qquad \forall x\, y.\ (x < y \vee y < x \vee x = y) \qquad \forall x\, y\, z.\ (x < y \wedge y < z \rightarrow x < z)$$

$$0 < 1 \qquad \forall x.\ (x + (-x) = 0) \qquad \forall x\, y\, z.\ (x < y \rightarrow x + z < y + z)$$

$$\forall x.\ \neg(0 < x \wedge x < 1) \qquad \forall x\, \exists y.\ \bigvee_{0 \leqslant r < n} x = ny + r$$

Theorem

- *$\mathbb{Z}$ with usual interpretations is model of LIA*
- *and it is unique model up to elementary equivalence*

i.e., same formulas hold

Example

- $x + y + z = 1 + 1 \wedge y < z \wedge -1 < y$ LIA-satisfiable, $v(x) = v(y) = 0$, $v(z) = 2$
- $x < 1 \wedge 1 < x + x$ LIA-unsatisfiable

Remarks

- LIA is also known as Presburger arithmetic: different but equivalent axiomatizations exist
- LIA has no multiplication: $x \cdot y$ and x^2 for variables x, y is not allowed

Syntactic Sugar

$\leqslant$	binary predicate	$s \leqslant t$ abbreviates $\neg(t < s)$
$>$ and $\geqslant$	binary predicates	use $s > t$ for $t < s$ and $s \geqslant t$ for $t \leqslant s$
$n\,\cdot$	unary functions $\forall n \in \mathbb{Z}$	$n \cdot t$ means $\underbrace{t + \ldots + t}_{n}$ if $n \geqslant 0$ $\underbrace{(-t) + \ldots + (-t)}_{n}$ if $n < 0$
n	constants $\forall n \in \mathbb{Z}$	n abbreviates $n \cdot 1$

Example (LIA with syntactic sugar)

- $x + y + z = 2 \ \wedge\ z > y \ \wedge\ y \geqslant 0$
- $x < 1 \ \wedge\ 2x > 1$
- $7x = 41$

Theorem

LIA is decidable and NP-complete

Definition (Theory of Linear Arithmetic over $\mathbb{Q}$ (LRA))

- signature
  - binary predicates $<$ and $=$
  - binary function $+$, unary function $-$, constants 0 and 1
  - unary (division) functions $(\_/n)$ for all $n > 1$
- axioms

$\forall x.\ (x = x)$	$\forall x\, y.\ (x = y \ \rightarrow\ y = x)$	$\forall x\, y\, z.\ (x = y \wedge y = z \ \rightarrow\ x = z)$
$\forall x.\ (x{+}0 = x)$	$\forall x\, y.\ (x{+}y = y{+}x)$	$\forall x\, y\, z.\ (x{+}(y{+}z) = (x{+}y){+}z)$
$\forall x.\ \neg(x < x)$	$\forall x\, y.\ (x < y \vee y < x \vee x = y)$	$\forall x\, y\, z.\ (x < y \wedge y < z \ \rightarrow\ x < z)$
$0 < 1$	$\forall x.\ (x + (-x) = 0)$	$\forall x\, y\, z.\ (x < y \ \rightarrow\ x{+}z < y{+}z)$
	$\forall x.\ (n \cdot (x/n) = x)$	for all $n > 1$

Theorem

- *$\mathbb{Q}$ with usual interpretations is model of LRA*
- *and it is the single unique model up to elementary equivalence*

Example

- $x{+}y{+}z = 1{+}1 \ \wedge\ y < z \ \wedge\ -1 < y$ LRA-satisfiable, $v(x) = v(y) = 0$, $v(z) = 2$
- $x < 1 \ \wedge\ 1 < x + x$ LRA-satisfiable with $v(x) = \frac{2}{3}$

Syntactic Sugar

use same shorthands as for LIA, plus

- $q\,\cdot$ unary functions $\forall q \in \mathbb{Q}$ $q \cdot t$ abbreviates $m \cdot t / n$ if $q = \frac{m}{n}$
- q constants $\forall q \in \mathbb{Q}$ q abbreviates $q \cdot 1$

Example (LRA with syntactic sugar)

- $\frac{4}{5}x = 2 \wedge \frac{x}{7} = \frac{y}{2} + 1$
- $x < \frac{7}{8} \wedge 2x > \frac{5}{4}$
- $7.5x = 41.2$

Theorem

LRA is decidable in polynomial time

Some History

1826 Fourier and Motzkin (1936) developed elimination algorithm for LRA

- takes doubly exponential time

1947 Dantzig proposed Simplex algorithm to solve optimization problem in LRA:

$$\text{maximize } c(\overline{x}) \qquad \text{such that} \qquad A\overline{x} \leqslant b \text{ and } \overline{x} \geqslant 0$$

for linear objective function c, matrix A, vector b, and vector of variables $\overline{x}$

- runs in exponential time, also known as linear programming

1960 Land and Doig: Branch-And-Bound to get LIA solution from LRA solution

1979 Khachiyan proposed polynomial Simplex based on ellipsoid method

1984 Karmakar proposed polynomial version based on interior points method

2000- SMT solvers use DPLL(T) version to solve satisfiability problem

$$A\overline{x} \leqslant b$$

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Aim

build theory solver for linear rational arithmetic (LRA):

decide whether set of linear (in)equalities is satisfiable over $\mathbb{Q}$

Disclaimer: Effects and Side Effects

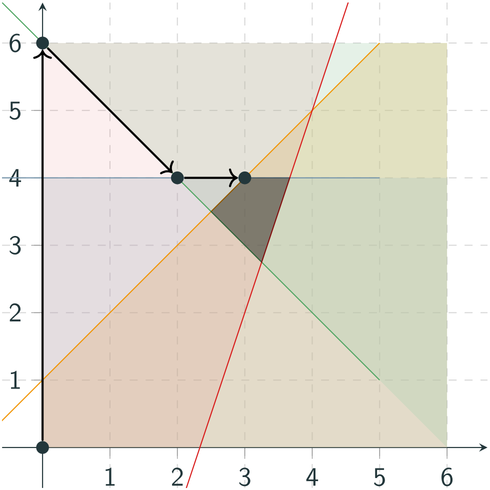
- guaranteed to solve all your real arithmetic problems
- consuming Simplex can cause initial dizzyness
- in some cases solving systems of linear inequalities can become addictive

Simplex, Visually

- constraints

  $$x - y \geqslant -1$$
  $$y \leqslant 4$$
  $$x + y \geqslant 6$$
  $$3x - y \leqslant 7$$

- solution space
- Simplex algorithm: improve assignment in 4 iterations
  - $x = 0,\ y = 0$
  - $x = 0,\ y = 6$
  - $x = 2,\ y = 4$
  - $x = 3,\ y = 4$



Definition (Problem in general form)

- variables $x_1, \dots, x_n$
- m equalities for $a_{ij} \in \mathbb{Q}$

$$\begin{aligned} a_{11}x_1 + \dots a_{1n}x_n &= 0 \\ \dots \\ a_{m1}x_1 + \dots a_{mn}x_n &= 0 \end{aligned}$$

- (optional) lower and upper bounds on variables for $l_i, u_i \in \mathbb{Q}$

$$l_i \leqslant x_i \leqslant u_i$$

no occurrences of $<$, $>$, or $\neq$

Lemma

set of LRA literals where all predicates are $\leqslant$, $\geqslant$, or $=$ can be turned into equisatisfiable general form

Example

$$\begin{aligned} x - y &\geqslant -1 \\ y &\leqslant 4 \\ x + y &\geqslant 6 \\ 3x - y &\leqslant 7 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} -x + y - s_1 &= 0 & s_1 &\leqslant 1 \\ y - s_2 &= 0 & s_2 &\leqslant 4 \\ -x - y - s_3 &= 0 & s_3 &\leqslant -6 \\ 3x - y - s_4 &= 0 & s_4 &\leqslant 7 \end{aligned}$$

slack variables

- s_1, s_2, s_3, s_4 are slack variables, x, y are problem variables

Representation

- represent equalities by $m \times (n+m)$ matrix A such that $A \cdot \begin{pmatrix} \overline{x} \\ \overline{s} \end{pmatrix} = 0$

$$\begin{aligned} -x + y - s_1 &= 0 & s_1 &\leqslant 1 \\ y - s_2 &= 0 & s_2 &\leqslant 4 \\ -x - y - s_3 &= 0 & s_3 &\leqslant -6 \\ 3x - y - s_4 &= 0 & s_4 &\leqslant 7 \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 3 & -1 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{aligned} s_1 &\leqslant 1 \\ s_2 &\leqslant 4 \\ s_3 &\leqslant -6 \\ s_4 &\leqslant 7 \end{aligned}$$

- simplified matrix presentation

$$\begin{array}{c} \\ \text{dependent variables} \rightarrow \end{array} \quad \begin{array}{c|cc} & x & y \\ \hline s_1 & -1 & 1 \\ s_2 & 0 & 1 \\ s_3 & -1 & -1 \\ s_4 & 3 & -1 \end{array} \quad \leftarrow \text{independent variables}$$

Notation

- simplified matrix is called tableau
- D is set of dependent (or basic) variables, in tableau listed on the left
- I is set of independent (or non-basic) variables, in tableau on top

DPLL(T) Simplex Algorithm

Input: conjunction of LRA literals φ without $<$, $>$, $\neq$

Output: satisfiable or unsatisfiable

1. transform φ into general form and construct tableau
2. fix order on variables and assign 0 to each variable
3. if all dependent variables satisfy their bounds then return satisfiable
4. otherwise, let $x \in D$ be variable that violates one of its bounds b
5. search for suitable variable $y \in I$ for pivoting with x
   (i.e., look for y whose value can be changed such that x is within b)
6. return unsatisfiable if no such variable exists
7. perform pivot operation on x and y
   (i.e., make x independent and y dependent)

9. improve assignment: set x to b, and update accordingly
10. go to step 3

Example

tableau	bounds	assignment

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} \end{array}$$

$s_1 \leqslant 1$
$s_2 \leqslant 4$
$s_3 \leqslant -6$
$s_4 \leqslant 7$

x	y	s_1	s_2	s_3	s_4
3	4	1	4	−7	5

1. Iteration 1
   - s_3 violates its bounds
   - decreasing s_3 requires to increase x or y because $s_3 = -x - y$: both suitable since they have no upper bound
   - pivot s_3 with y:

$$y = -x - s_3 \qquad s_1 = -2x - s_3$$
$$s_2 = -x - s_3 \qquad s_4 = 4x + s_3$$

   - update assignment: set s_3 to violated bound -6 and propagate

$$s_3 = -6 \qquad y = 6$$
$$s_1 = 6 \qquad s_2 = 6 \qquad s_4 = -6$$

Simplex, Visually

- constraints

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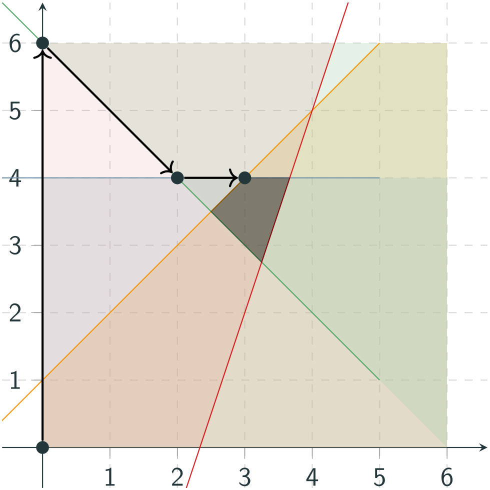
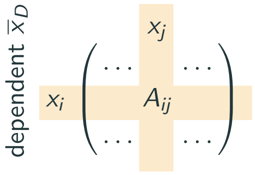


DPLL(T) Simplex Algorithm



$$A\overline{x}_I = \overline{x}_D \qquad (1)$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \qquad (2)$$

Invariant

- (1) is satisfied and (2) holds for all independent variables

Pivoting

- swap dependent x_i and independent x_j, so $x_i \in D$ and $x_j \in I$

$$x_i = \sum_{x_k \in I} A_{ik} x_k \quad \Longrightarrow \quad x_j = \underbrace{\frac{1}{A_{ij}}(x_i - \sum_{x_k \in I - \{x_j\}} A_{ik} x_k)}_{t} \qquad (\star)$$

new row

updated other rows

- new tableau A' consists of $(\star)$ and $x_m = A_{mj} t + \sum_{x_k \in I - \{x_j\}} A_{mk} x_k \quad \forall x_m \in D - \{x_i\}$

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of x_k is updated using A' for all $\forall x_m \in D - \{x_i\}$

DPLL(T) Simplex Algorithm

$$A\overline{x}_I = \overline{x}_D \qquad (1)$$

$$-\infty \leqslant l_i \leqslant x_i \leqslant u_i \leqslant +\infty \qquad (2)$$

Suitable pivot variable

- suppose dependent variable x_i violates lower and/or upper bound
- then x_j is suitable for pivoting with x_i if
  - if $x_i < l_i$: $(A_{ij} > 0$ and $x_j < u_j)$ or $(A_{ij} < 0$ and $x_j > l_j)$
  - if $x_i > u_i$: $(A_{ij} > 0$ and $x_j > l_j)$ or $(A_{ij} < 0$ and $x_j < u_j)$

want to increase x_i | need to increase x_j | need to decrease x_j

want to decrease x_i | need to decrease x_j | need to increase x_j

Observation

selecting variables and pivots in unfortunate order may lead to non-termination

Bland's rule

select variable x_i in step 4 and x_j in step 5 such that (x_i, x_j) is minimal with respect to lexicographic extension of order on variables

Lemma

- *Simplex terminates if pivot variables are selected according to Bland's rule*
- *problem is satisfiable iff Simplex returns satisfiable*

How to Deal With Strict Inequalities?

replace in LRA formula φ every strict inequality

$$a_1x_1 + \cdots + a_nx_n < b$$

by non-strict inequality

$$a_1x_1 + \cdots + a_nx_n \leqslant b - \delta$$

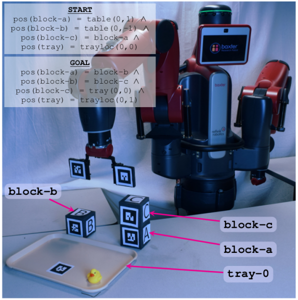
to obtain formula φ_δ in LRA without $<$, and treat δ as variable during Simplex

Lemma

φ *is satisfiable* $\iff$ $\exists$ *rational number* $\delta > 0$ *such that* φ_δ *is satisfiable*

Application: Motion Planning for Robots

- robots needs to plan motions
  to place objects correctly
- instance of *constraint based planning*
- encoding
  - fix number of time slots $t_1, \ldots, t_n$
  - action variable a_i for time t_i encodes which action performed at time t_i (one action per time)
  - actions require precondition and imply postcondition
  - use arithmetic to minimize path



Neil T. Dantam, Zachary K. Kingston, Swarat Chaudhuri, and Lydia E. Kavraki.
Incremental Task and Motion Planning: A Constraint-Based Approach.
In: The International Journal of Robotics Research, 2018.

(Almost) Everything is Better With Arithmetic

LRA and LIA admit more efficient encodings of

- n-queens
- Sudoku
- graph coloring
- Minesweeper
- travelling salesperson
- rabbit problem
- planning problems
- scheduling problems
- component configuration problems
- everything with cardinality constraints
- ...

Bibliography

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A Fast Linear-Arithmetic Solver for DPLL(T).
In Proc. of International Conference on Computer Aided Verification, pp. 81–94, 2006.

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Integrating Simplex with DPLL(T)
Technical Report SRI–CSL–06–01, SRI International, 2006

Test on December 2

- 50 minutes
- open (paper) book: bring arbitrary amount of printed paper, but use no electronic devices
- questions are like homework exercises:
  e.g., DPLL, implication graphs, give minimal unsatisfiable core of formula, equality graphs, congruence closure, DPLL(T), ... (no Simplex)