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SAT and SMT Solving

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Outline

- Summary of Last Week
- Cyclic Simplex Example
- Branch and Bound
- Fourier-Motzkin Elimination

Definition (Theory of Linear Arithmetic over C)

- for variables $x_1, \ldots, x_n$, formulas built according to grammar

$$\varphi ::= \varphi \wedge \varphi \mid t = t \mid t < t \mid t \leqslant t$$

$$t ::= a_1x_1 + \cdots + a_nx_n + b \qquad\qquad \text{for } a_1, \ldots, a_n, b \in \text{ in carrier } C$$

- axioms are equality axioms plus calculation rules of arithmetic over C
- solution assigns values in C to $x_1, \ldots, x_n$

Definitions

- carrier $\mathbb{Q}$: linear real arithmetic (LRA),
  DPLL(T) simplex algorithm is decision procedure
- carrier $\mathbb{Z}$: linear integer arithmetic (LIA)

DPLL(T) Simplex Algorithm (1)

- linear arithmetic constraint solving over real or rational variables
- $x_1, \ldots, x_n$ are split into dependent variables $\overline{x}_D$ and independent variables $\overline{x}_I$

Input

constraints plus upper and lower bounds for $x_1, \ldots, x_n$:

$$A\,\overline{x}_I = \overline{x}_D \qquad \text{with tableau } A \in \mathbb{Q}^{|D| \times |I|} \tag{1}$$

$$l_i \leqslant x_i \leqslant u_i \tag{2}$$

Output

satisfying assignment or "unsatisfiable"

Invariant

(1) is satisfied and (2) holds for all independent variables x_i

DPLL(T) Simplex Algorithm (2)

$$A\overline{x}_I = \overline{x}_D \tag{1}$$

$$l_i \leqslant x_i \leqslant u_i \tag{2}$$

Method

- if (2) holds for all dependent variables, return current assignment
- otherwise select dependent variable $x_i \in D$ which violates (2)
- select suitable independent variable $x_j \in I$ such that x_i and x_j can be swapped in a pivoting step, resulting in new tableau

  $$A'\, x_{I'} = x_{D'}$$

  with $I' = I \cup \{x_i\} - \{x_j\}$ and $D' = D \cup \{x_j\} - \{x_i\}$
- change value of x_i to l_i or u_i, update values of dependent variables accordingly

DPLL(T) Simplex Algorithm (3)

$$A\overline{x}_I = \overline{x}_D \tag{1}$$

$$l_i \leqslant x_i \leqslant u_i \tag{2}$$

Pivoting

- swap dependent x_i and non-dependent x_j

$$x_i = \sum_{x_k \in I} A_{ik} x_k \qquad \Longrightarrow \qquad x_j = \underbrace{\frac{1}{A_{ij}}(x_i - \sum_{x_k \in I - \{x_j\}} A_{ik} x_k)}_{t} \tag{$\star$}$$

- new tableau A' consists of ($\star$) and $x_m = A_{mj} t + \sum_{x_k \in I - \{x_j\}} A_{mk} x_k \;\; \forall x_m \in D - \{x_i\}$

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of x_k is updated using A' for all $\forall x_k \in D - \{x_i\}$

DPLL(T) Simplex Algorithm (4)

$$A\overline{x}_I = \overline{x}_D \tag{1}$$

$$l_i \leqslant x_i \leqslant u_i \tag{2}$$

Suitability

- dependent variable x_i violates lower and/or upper bound
- pick independent variable x_j such that
  - if $x_i < l_i$: $A_{ij} > 0$ and $x_j < u_j$ or $A_{ij} < 0$ and $x_j > l_j$
  - if $x_i > u_i$: $A_{ij} > 0$ and $x_j > l_j$ or $A_{ij} < 0$ and $x_j < u_j$
- problem is unsatisfiable if no suitable pivot exists

Bland's Rule

- pick lexicographically smallest (i, j) that is suitable pivot
- guarantees termination

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Example

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_3 \\ x_4 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 0 & 0 \end{array} \quad \longleftarrow \quad \begin{array}{cc} & \begin{array}{cc} x_1 & x_4 \end{array} \\ \begin{array}{c} x_3 \\ x_2 \end{array} & \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 1 & 2 & 1 \end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \uparrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \end{array} & \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline \frac{7}{3} & -\frac{11}{3} & -5 & 1 \end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \uparrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -\frac{10}{3} & -\frac{1}{3} & -4 & -7 \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \end{array} & \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 3 & -4 & -5 & 2 \end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \uparrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_4 \end{array} \\ \begin{array}{c} x_3 \\ x_2 \end{array} & \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -5 & -11 & -7 \end{array} \quad \longrightarrow \quad \begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_3 \\ x_4 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -4 & -9 & -6 \end{array}$$

Example

$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_3 \\ x_4 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \end{array} & \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -4 & 0 & -4 & -8 \end{array}$$

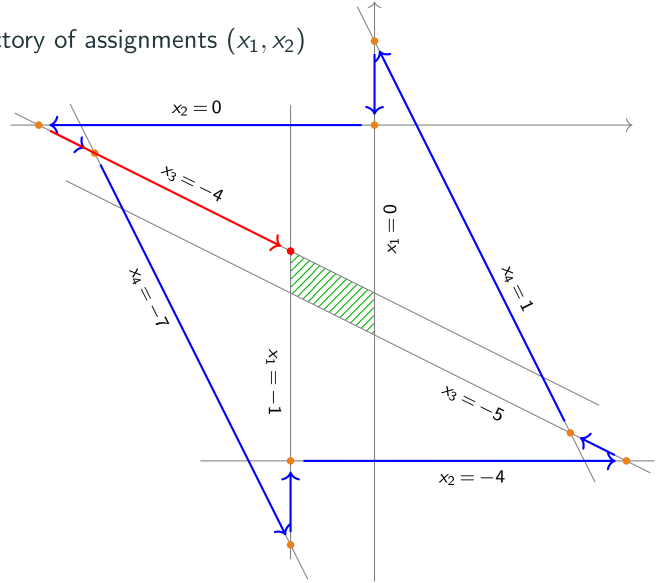
violation of Bland's rule

$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} x_3 & x_1 \end{array} \\ \begin{array}{c} x_2 \\ x_4 \end{array} & \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline -1 & -\frac{3}{2} & -4 & -\frac{7}{2} \end{array}$$

satisfying assignment

trajectory of assignments (x_1, x_2)



Disclaimer

if advice of Dr. Bland is neglected, no cure is guaranteed!

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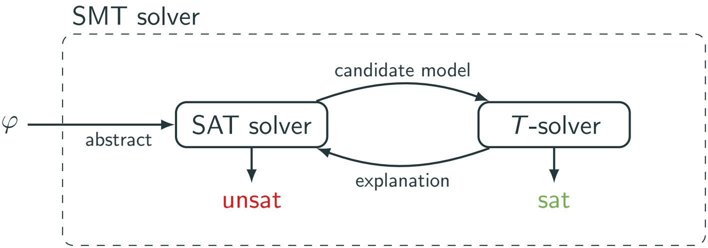
How to Be Lazy

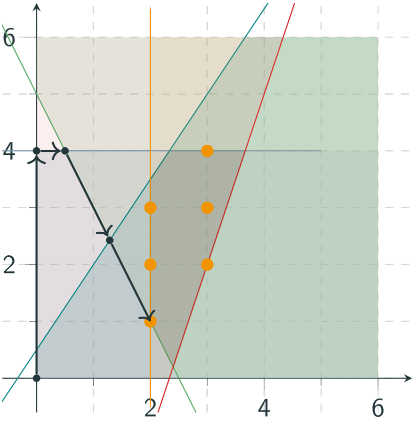


Theory T	**T-solving method**	
▶ equality logic	equality graphs	✓
▶ equality + uninterpreted functions (EUF)	congruence closure	✓
▶ linear real arithmetic (LRA)	DPLL(T) Simplex	✓
▶ linear integer arithmetic (LIA)	DPLL(T) Simplex + cuts	
▶ bitvectors (BV)		
▶ arrays (A)		

Example

$$3x - 2y \geqslant -1$$
$$y \leqslant 4$$
$$2x + y \geqslant 5$$
$$3x - y \leqslant 7$$

- looking for solution in $\mathbb{Z}^2$
- infinite $\mathbb{Q}^2$ solution space, six solutions in $\mathbb{Z}^2$
- Simplex returns $(\frac{9}{7}, \frac{17}{7})$



Idea (Branch and Bound)

- add constraints that exclude solution in $\mathbb{Q}^2$ but do not change solutions in $\mathbb{Z}^2$
- in current solution $1 < x < 2$, so use Simplex on two augmented problems:
  - $C \wedge x \leqslant 1$ unsatisfiable
  - $C \wedge x \geqslant 2$ satisfiable, Simplex can return $(2, 1)$

Algorithm BranchAndBound(φ)

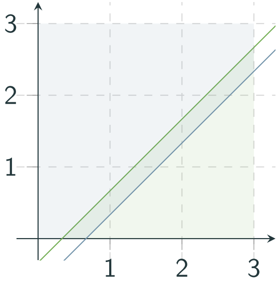
Input: LIA constraint φ
Output: unsatisfiable, or satisfying assignment

```
S ← decide φ over ℚ                                   ▷ e.g. by Simplex
if S' = unsatisfiable then
    return unsatisfiable
else if S is solution over ℤ then
    return S
else
    x ← variable assigned non-integer value q in S
    S' = BranchAndBound(φ ∧ x ⩽ ⌊q⌋)
    if S' ≠ unsatisfiable then
        return S'
    else
        return BranchAndBound(φ ∧ x ⩾ ⌈q⌉)
```

Definition

$\mathbb{Q}^2$-solution space of linear arithmetic problem $Ax \leqslant b$ is bounded
if for all x_i there exist $l_i, u_i \in \mathbb{Q}$ such that all $\mathbb{Q}^2$-solutions v satisfy $l_i \leqslant v(x_i) \leqslant u_i$

Example



- $3x - 3y \geqslant 1 \land 3x - 3y \leqslant 2$
- unbounded problem
- no solution in $\mathbb{Z}^2$
- BranchAndBound keeps adding $x \geqslant n,\ y \geqslant m$
  $x \leqslant n,\ y \leqslant m$

Remarks

- BranchAndBound might not terminate if solution space is unbounded
- methods exist to derive solution bounds from tableau, but bounds are often too high for efficient practical procedures
- use cutting planes to restrict solution space more efficiently

LIA Application: Finding Work Schedules

Example (Scheduling Problem)

Is there a six-week cyclic work schedule for 22 employees who work 8-hour shifts, 24/7, 5 working days per week such that the following holds: In morning and afternoon shifts 6 employees are present, in night shifts 3. Joe does only morning shifts, Sally does not work on Sundays. Nobody works more than 6 days in a row.

Shift Schedule Requirements

- number of employees n
- set of shifts A (activities to be distributed)
- length of schedule (e.g. one week) and cyclicity
- requirement matrix R: R_{ij} is # employees required in shift i of day j
- prohibited shift sequences, maximal length of work blocks, ...

LIA Encoding

- integer variable corresponding to employee for each activity
- cardinality constraints for requirement matrix
- ...

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Fourier-Motzkin Elimination

Aim

build theory solver for linear rational arithmetic (LRA):
decide whether conjunction of linear (in)equalities φ is satisfiable over $\mathbb{Q}$

Preprocessing: eliminate $\neq$

$(t_1 \neq t_2) \wedge \varphi$ is satisfiable iff $(t_1 < t_2) \wedge \varphi$ or $(t_1 > t_2) \wedge \varphi$ are satisfiable

Definition (Elimination step)

- for variable x in φ, can write φ as

  $$\bigwedge_i (x < U_i) \;\wedge\; \bigwedge_j (x \leqslant u_j) \;\wedge\; \bigwedge_k (L_k < x) \;\wedge\; \bigwedge_m (\ell_m \leqslant x) \;\wedge\; \psi$$

  where U_i, u_j, L_k, ℓ_m, ψ are without x

  formula without x

- let *elim*(φ, x) be conjunction of

  $$\bigwedge_i \bigwedge_k (L_k < U_i) \quad \bigwedge_i \bigwedge_m (\ell_m < U_i) \quad \bigwedge_j \bigwedge_k (L_k < u_j) \quad \bigwedge_j \bigwedge_m (\ell_m \leqslant u_j) \quad \psi$$

Lemma

φ is LRA-satisfiable iff elim(φ, x) is LRA-satisfiable

Observation

- can subsequently eliminate all variables
- checking satisfiability of formula without variables is easy
- so obtain decision procedure for LRA!

Example

… on blackboard

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