universität innsbruck

SAT and SMT Solving

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lecture 9
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Outline

- Summary of Last Week
- Cutting Planes
- Bounds for Integer Solutions

Idea (Branch and Bound)

- given $\mathbb{Q}^2$ solution α, add constraints to exclude α but preserve $\mathbb{Z}^2$ solutions: if $a < \alpha(x) < a_1$, use Simplex on problems $C \wedge x \leqslant a$ and $C \wedge x \geqslant a + 1$
- need not terminate if solution space is unbounded

Algorithm BranchAndBound(φ)

Input: LIA constraint φ
Output: unsatisfiable, or satisfying assignment

let *res* be result of deciding φ over $\mathbb{Q}$ ▷ e.g. by Simplex
if *res* is unsatisfiable **then**
    return unsatisfiable
else if *res* is solution over $\mathbb{Z}$ **then**
    return *res*
else
    let x be variable assigned non-integer value q in *res*
    $res = \text{BranchAndBound}(\varphi \wedge x \leqslant \lfloor q \rfloor)$
    return $res \neq$ unsatisfiable **?** *res* **:** $\text{BranchAndBound}(\varphi \wedge x \geqslant \lceil q \rceil)$

Definition

$\mathbb{Q}^2$-solution space of linear arithmetic problem $Ax \leqslant b$ is bounded if for all x_i there exist $l_i, u_i \in \mathbb{Q}$ such that all $\mathbb{Q}^2$-solutions v satisfy $l_i \leqslant v(x_i) \leqslant u_i$

Theorem

If solution space to φ is bounded then BranchAndBound(φ) returns unsatisfiable iff φ has no solution in $\mathbb{Z}^2$

Fourier-Motzkin Elimination

Aim

build theory solver for linear rational arithmetic (LRA):
decide whether conjunction of linear (in)equalities φ is satisfiable over $\mathbb{Q}$

Preprocessing: eliminate $\neq$

$(t_1 \neq t_2) \wedge \varphi$ is satisfiable iff $(t_1 < t_2) \wedge \varphi$ or $(t_1 > t_2) \wedge \varphi$ are satisfiable

Definition (Elimination step)

- for variable x in φ, can write φ as

$$\bigwedge_i (x < U_i) \;\wedge\; \bigwedge_j (x \leqslant u_j) \;\wedge\; \bigwedge_k (L_k < x) \;\wedge\; \bigwedge_m (\ell_m \leqslant x) \;\wedge\; \psi$$

  where U_i, u_j, L_k, ℓ_m, ψ are without x

- let $elim(\varphi, x)$ be conjunction of

$$\bigwedge_i \bigwedge_k (L_k < U_i) \quad \bigwedge_i \bigwedge_m (\ell_m < U_i) \quad \bigwedge_j \bigwedge_k (L_k < u_j) \quad \bigwedge_j \bigwedge_m (\ell_m \leqslant u_j) \quad \psi$$

Lemma

φ is LRA-satisfiable iff $elim(\varphi, x)$ is LRA-satisfiable

Observation

- can subsequently eliminate all variables
- checking satisfiability of formula without variables is easy
- so obtain decision procedure for LRA!

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Remark

worst-case complexity of FME is double exponential in number of variables

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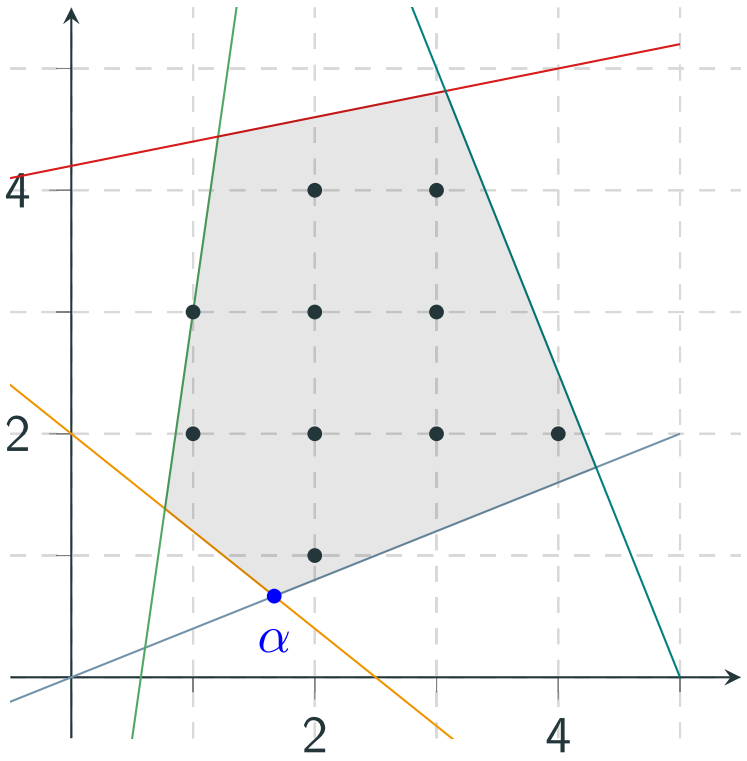
Consider set of constraints over linear integer arithmetic.

Example



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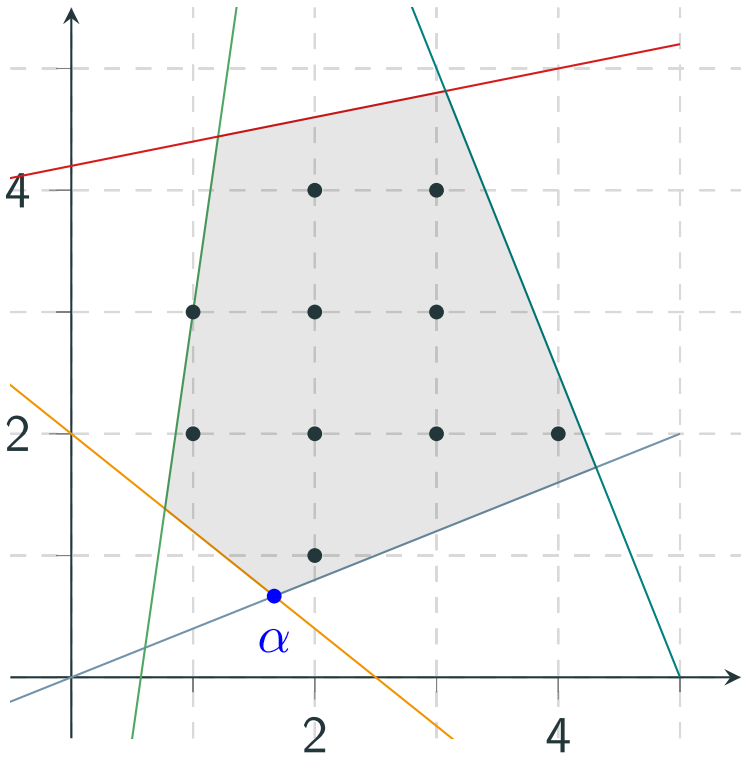

Definition (Cut)

given solution α over $\mathbb{Q}^n$, cut is inequality $a_1x_1 + \cdots + a_nx_n \leqslant b$ which is not satisfied by α but by every $\mathbb{Z}^n$-solution

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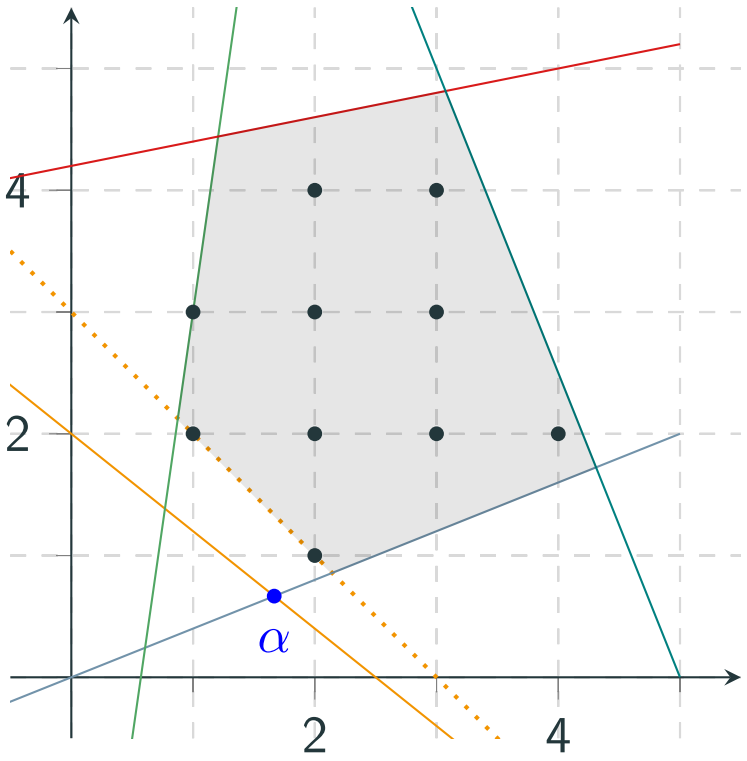

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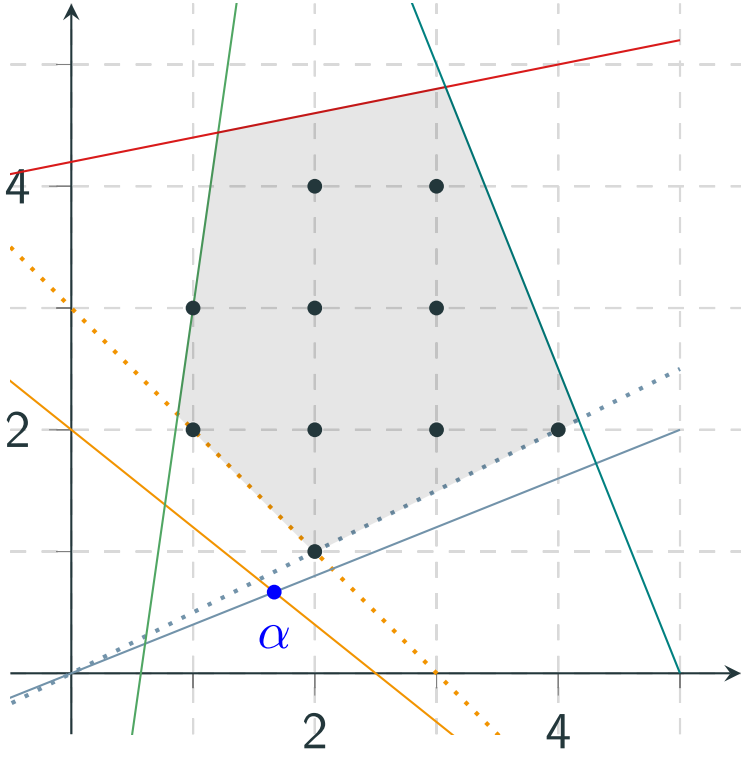

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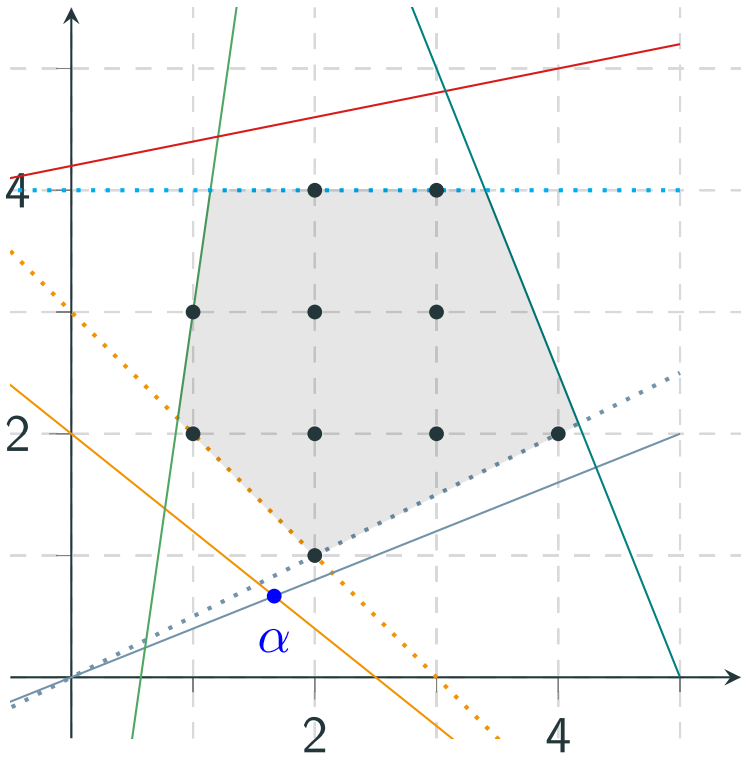

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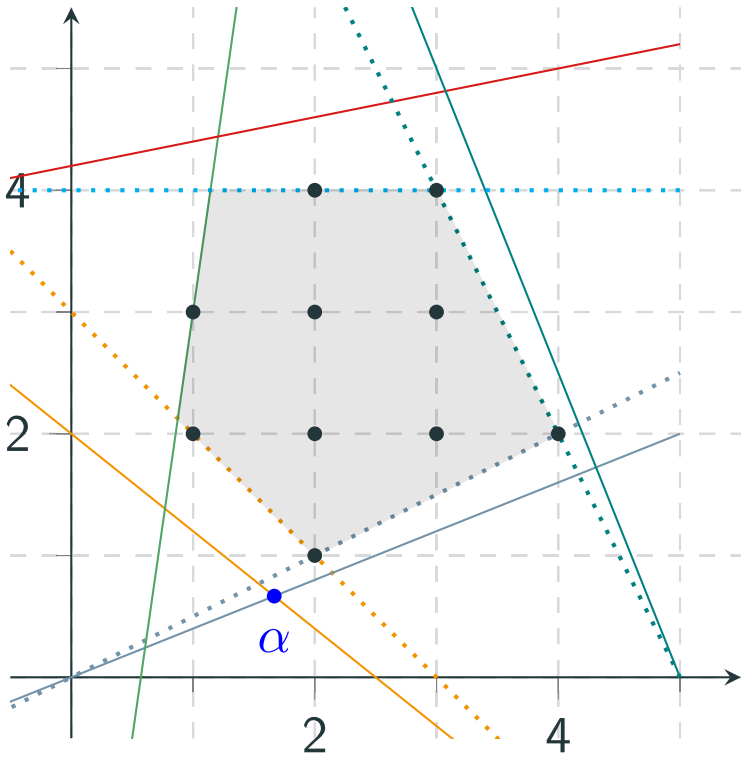

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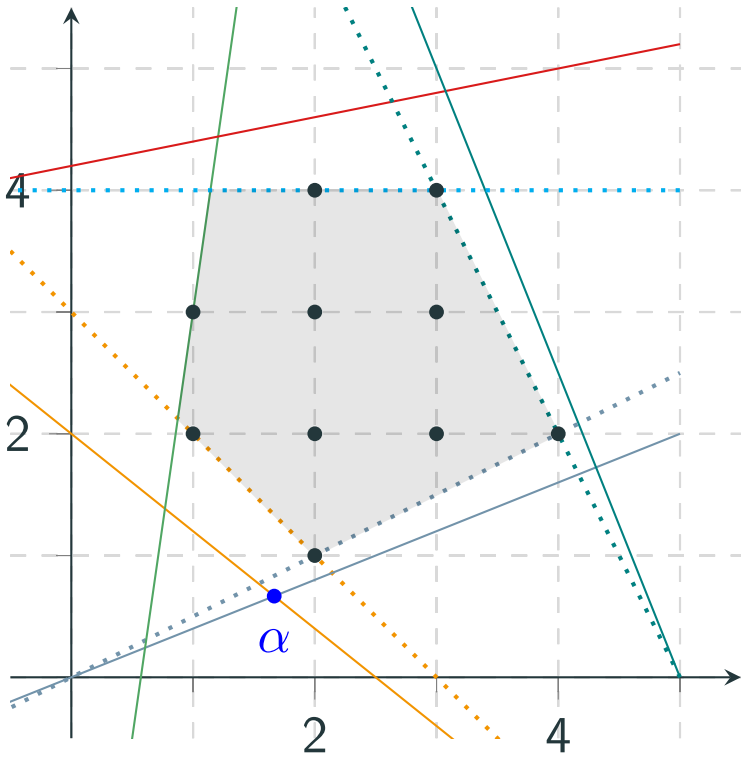

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like in BranchAndBound, keep adding cuts until integer solution found

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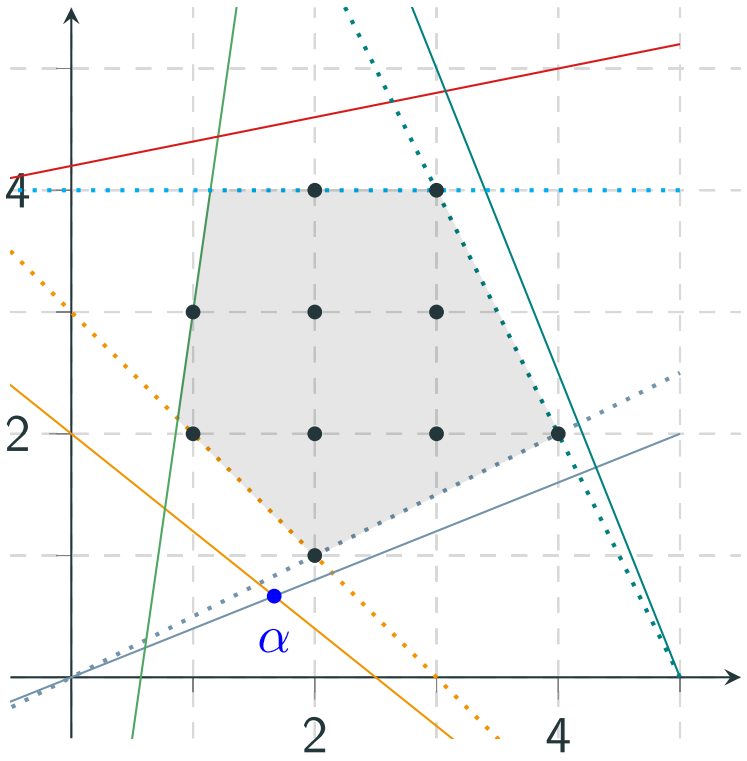

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need not terminate
for unbounded problems

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Gomory Cuts: Assumptions

- Simplex returned solution α over $\mathbb{Q}^n$:
  final tableau is A with dependent variables D and independent variables I

$$A\overline{x}_I = \overline{x}_D \tag{1}$$

$$l_k \leqslant x_k \leqslant u_k \quad \forall x_k \tag{2}$$

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Lemma (Gomory Cut)

the following inequality is a cut:

$$\sum_{x_j \in L^+} \frac{A_{ij}}{1-c}(x_j - l_j) - \sum_{x_j \in U^-} \frac{A_{ij}}{1-c}(u_j - x_j) - \sum_{x_j \in L^-} \frac{A_{ij}}{c}(x_j - l_j) + \sum_{x_j \in U^+} \frac{A_{ij}}{c}(u_j - x_j) \geqslant 1$$

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not satisfied by α: terms $x_j - l_j$ and $u_j - x_j$ evaluate to 0

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$$A\overline{x}_I = \overline{x}_D \tag{1}$$

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Proof (1)

- set up conditions for integer solution $\overline{x}$ to (1) and (2)

$$A\overline{x}_I = \overline{x}_D \tag{1}$$

$$l_k \leqslant x_k \leqslant u_k \quad \forall x_k \tag{2}$$

Proof (1)

- set up conditions for integer solution $\overline{x}$ to (1) and (2)
- $\overline{x}$ satisfies i-th row of (1):

$$x_i = \sum_{x_j \in I} A_{ij} x_j \tag{3}$$

$$A\overline{x}_I = \overline{x}_D \tag{1}$$

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- because α is solution, it holds that

$$\alpha(x_i) = \sum_{x_j \in I} A_{ij} \alpha(x_j) \tag{4}$$

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- set up conditions for integer solution $\overline{x}$ to (1) and (2)
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- subtract (4) from (3):

$$x_i - \alpha(x_i) = \sum_{x_j \in I} A_{ij}(x_j - \alpha(x_j)) \tag{5}$$

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Proof (1)

- set up conditions for integer solution $\overline{x}$ to (1) and (2)
- $\overline{x}$ satisfies i-th row of (1):

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- because α is solution, it holds that

$$\alpha(x_i) = \sum_{x_j \in I} A_{ij} \alpha(x_j) \tag{4}$$

- subtract (4) from (3):

$$\begin{aligned} x_i - \alpha(x_i) &= \sum_{x_j \in I} A_{ij}(x_j - \alpha(x_j)) \\ &= \sum_{x_j \in L} A_{ij}(x_j - l_j) - \sum_{x_j \in U} A_{ij}(u_j - x_j) \end{aligned} \tag{5}$$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$, so

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

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- for integer solution $\overline{x}$ left-hand side must be integer, so also right-hand side

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- for integer solution $\overline{x}$ left-hand side must be integer, so also right-hand side
- abbreviate

$$\mathcal{L}^+ = \sum_{x_j \in L^+} A_{ij}(x_j - l_j)$$
$$\mathcal{L}^- = \sum_{x_j \in L^-} A_{ij}(x_j - l_j)$$

so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

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so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

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  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

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  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

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  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$,

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

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  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$, $\mathcal{U}^- \leqslant 0$

Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

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  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$

- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$, $\mathcal{U}^- \leqslant 0$
- distinguish $\mathcal{L} \geqslant \mathcal{U}$ or $\mathcal{L} < \mathcal{U}$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$

since $\mathcal{L}^+ \geqslant \mathcal{L}$ and $\mathcal{U}^- \leqslant \mathcal{U}$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
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  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- otherwise $\mathcal{L} < \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- otherwise $\mathcal{L} < \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
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- otherwise $\mathcal{L} < \mathcal{U}$:
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  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$

since $\mathcal{U}^+ \geqslant \mathcal{U}$ and $\mathcal{L}^- \leqslant \mathcal{L}$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- otherwise $\mathcal{L} < \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
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- otherwise $\mathcal{L} < \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$
- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as well as c and $1 - c$

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
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  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$
- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as well as c and $1 - c$
- add (7) and (8) to obtain cut

$$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) + \frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1$$

■

Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) \geqslant 1 \tag{7}$$
- otherwise $\mathcal{L} < \mathcal{U}$:
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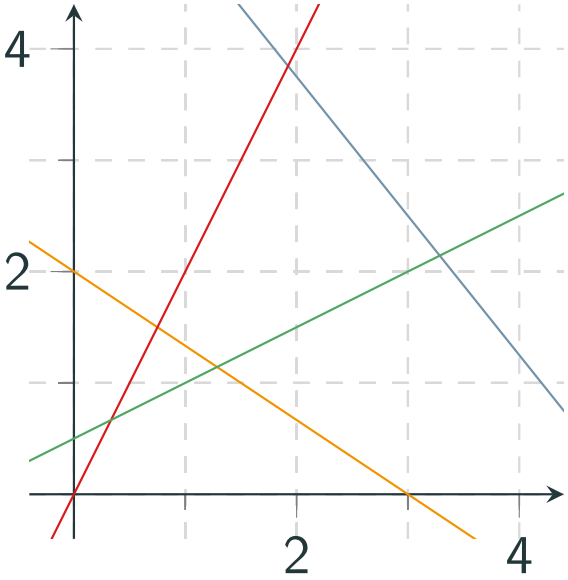
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1 \tag{8}$$
- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as
- add (7) and (8) to obtain cut

$$\frac{1}{1-c}\left(\mathcal{L}^+ - \mathcal{U}^-\right) + \frac{1}{c}\left(\mathcal{U}^+ - \mathcal{L}^-\right) \geqslant 1$$

the desired monster inequality!

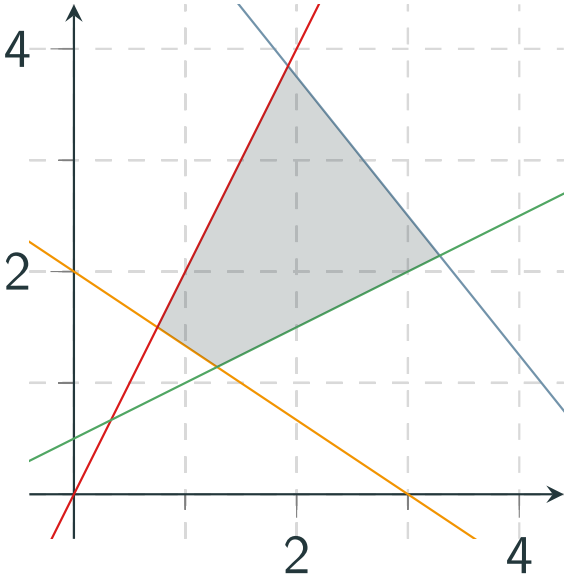
■

Example



$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

Example



$$-2x - 3y \leqslant -6$$
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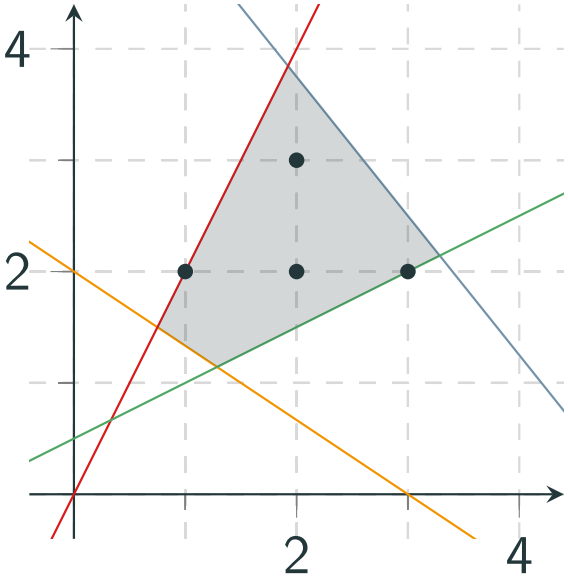
- infinite $\mathbb{Q}^2$-solution space

Example



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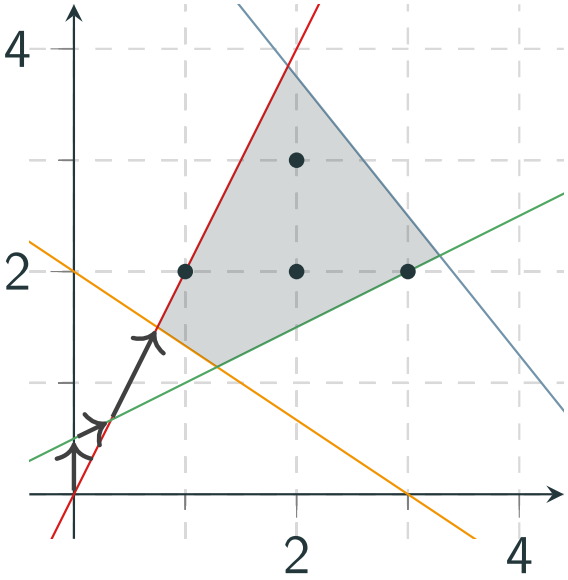
- infinite $\mathbb{Q}^2$-solution space
- four solutions in $\mathbb{Z}^2$

Example



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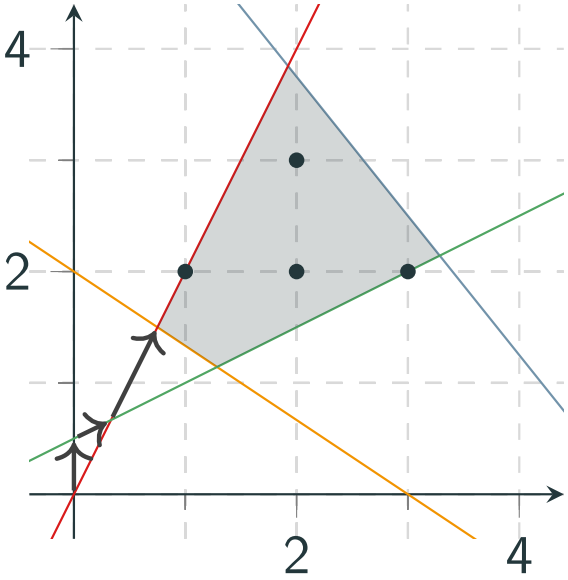
- infinite $\mathbb{Q}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

Example



$$-2x - 3y \leqslant -6$$
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- infinite $\mathbb{Q}^2$-solution space
- four solutions in $\mathbb{Z}^2$
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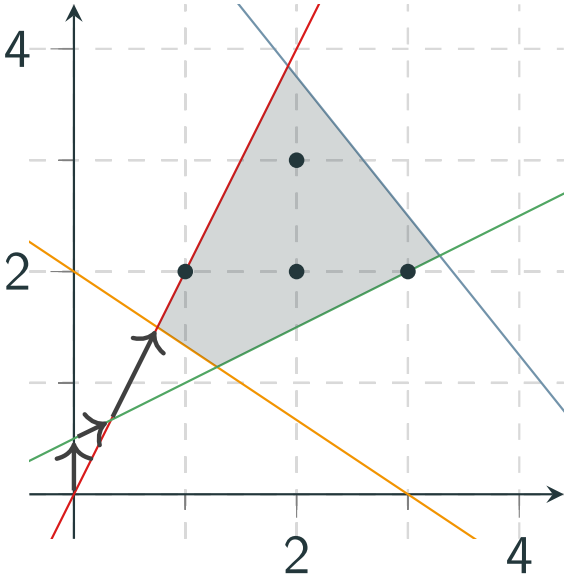
initial tableau

Example



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solution

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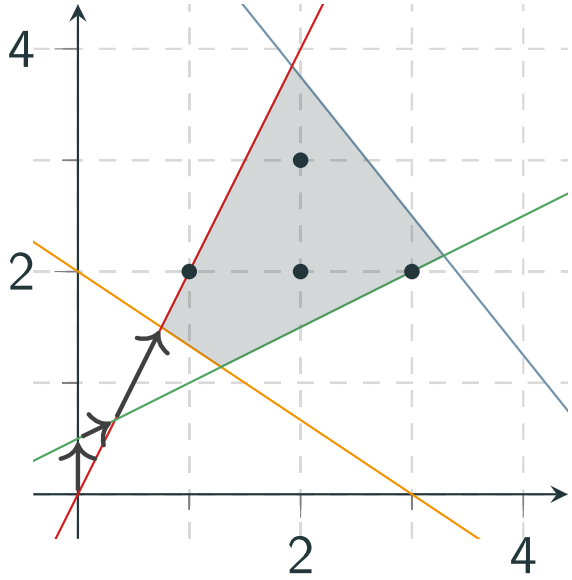

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final tableau solution

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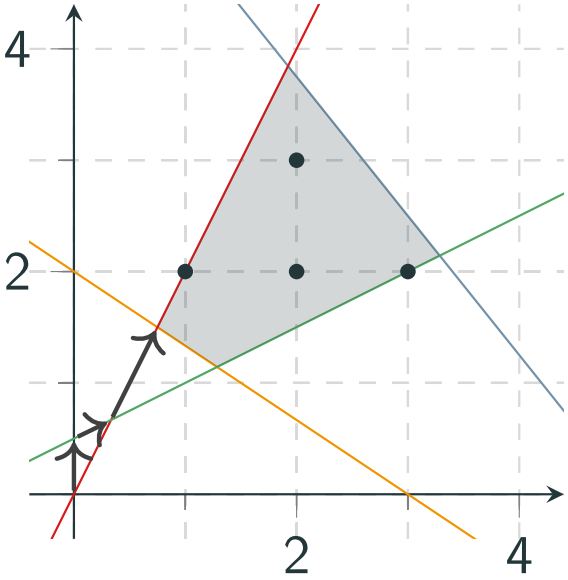

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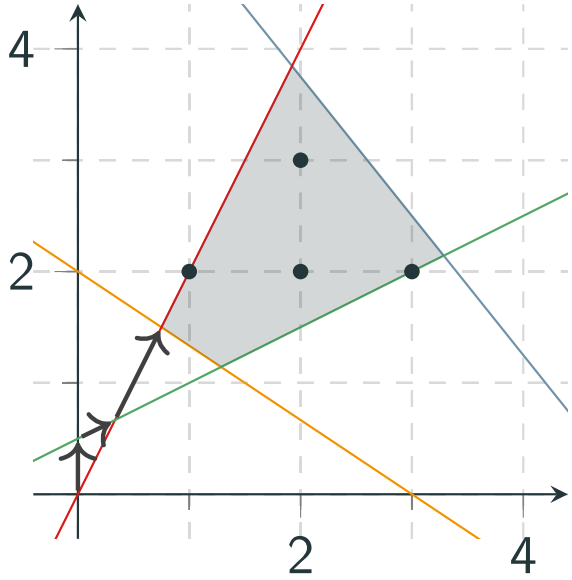

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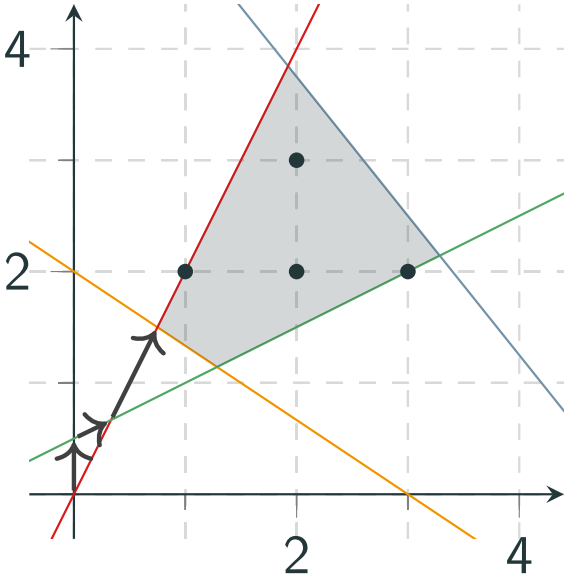

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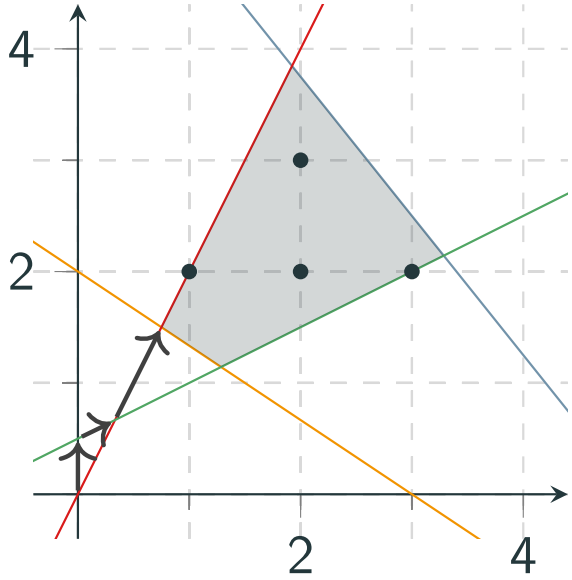

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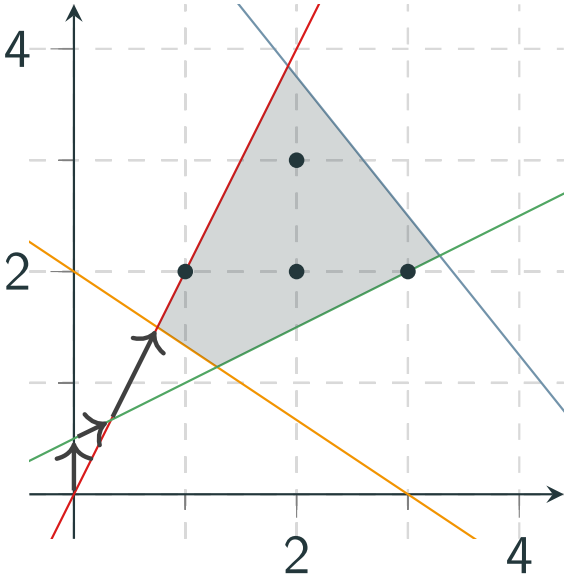

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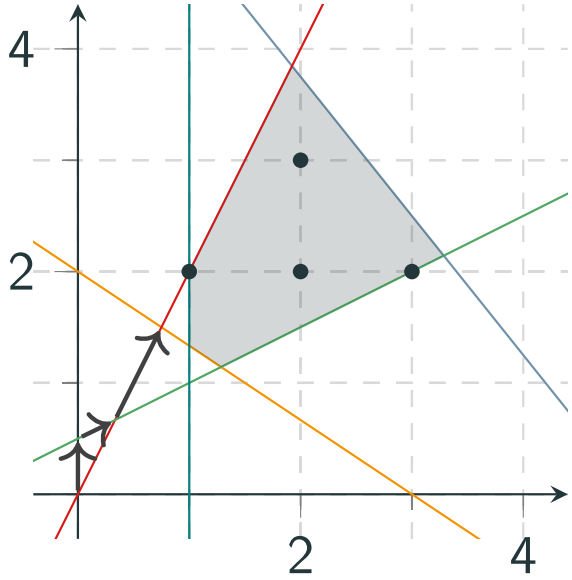

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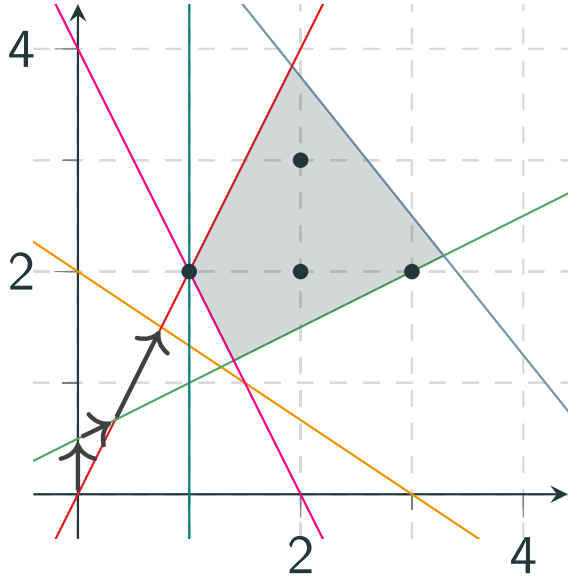

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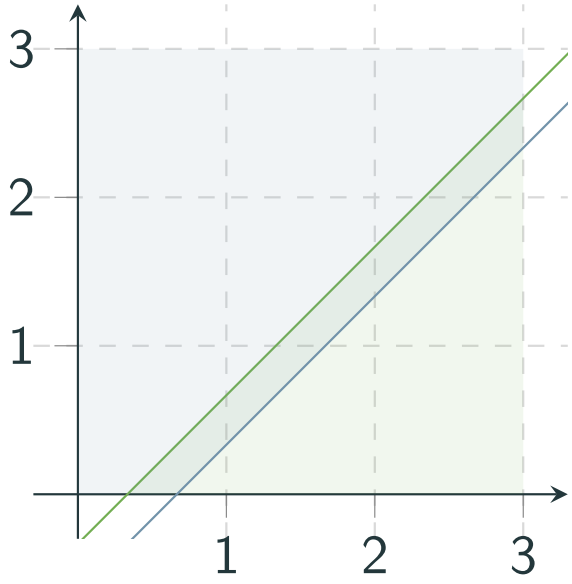
Outline

- Summary of Last Week
- Cutting Planes
- Bounds for Integer Solutions

Example



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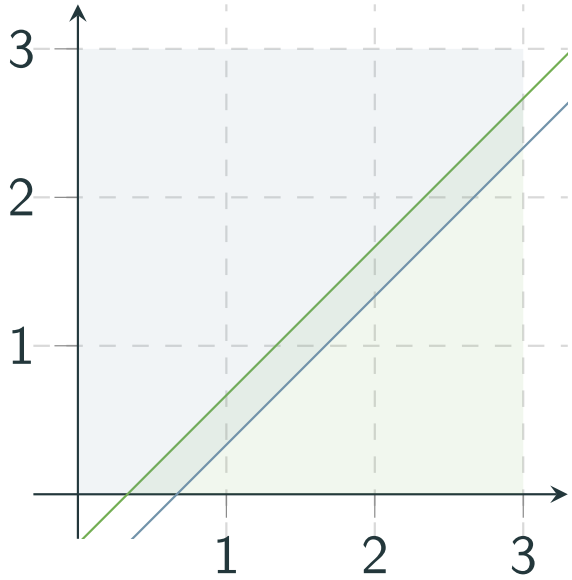

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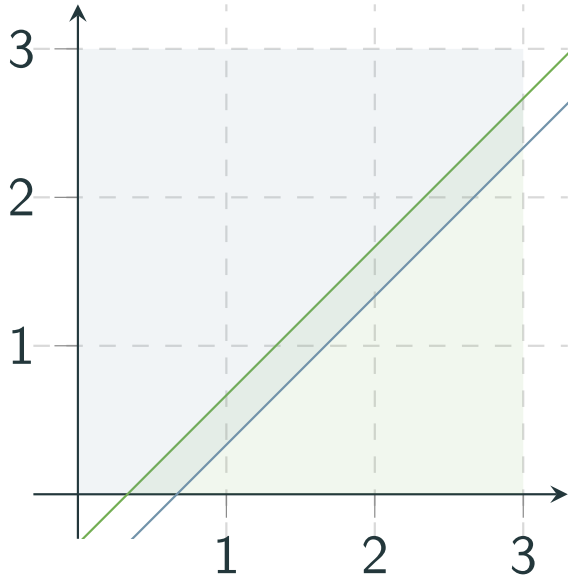

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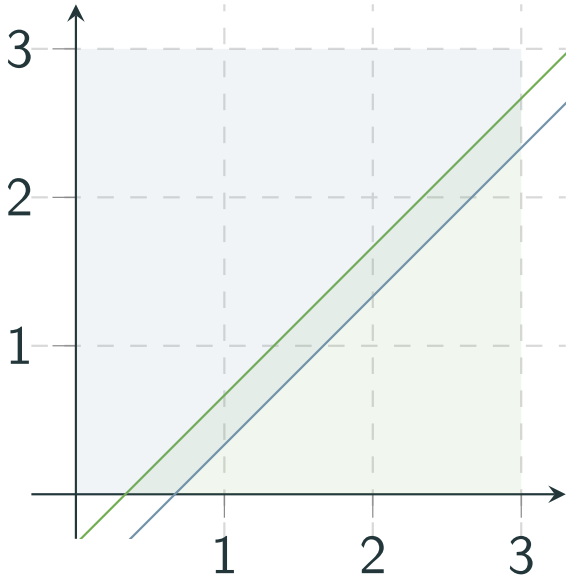

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(material in the remainder of this section is by René Thiemann)

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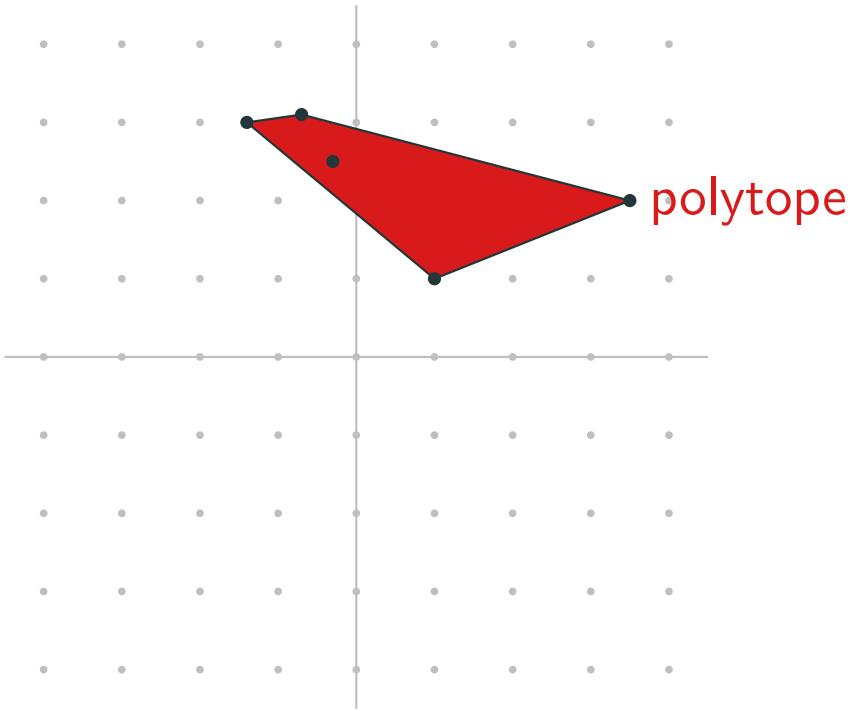

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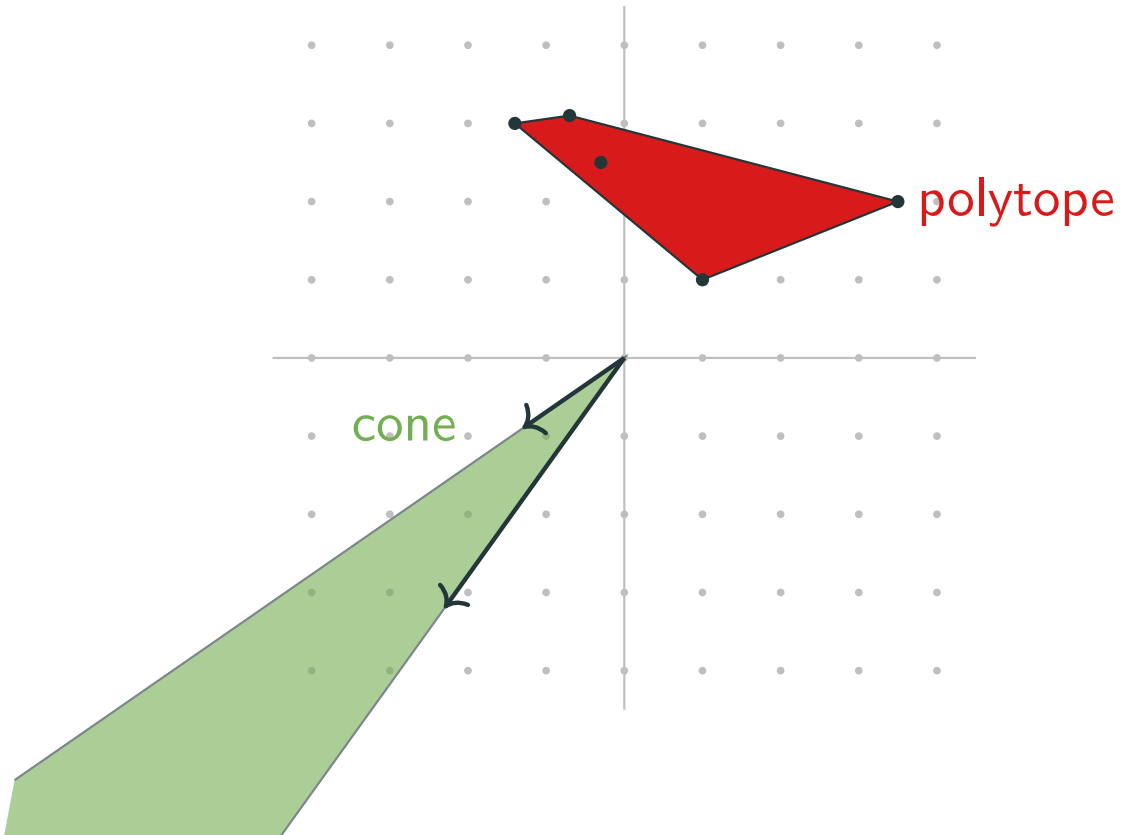

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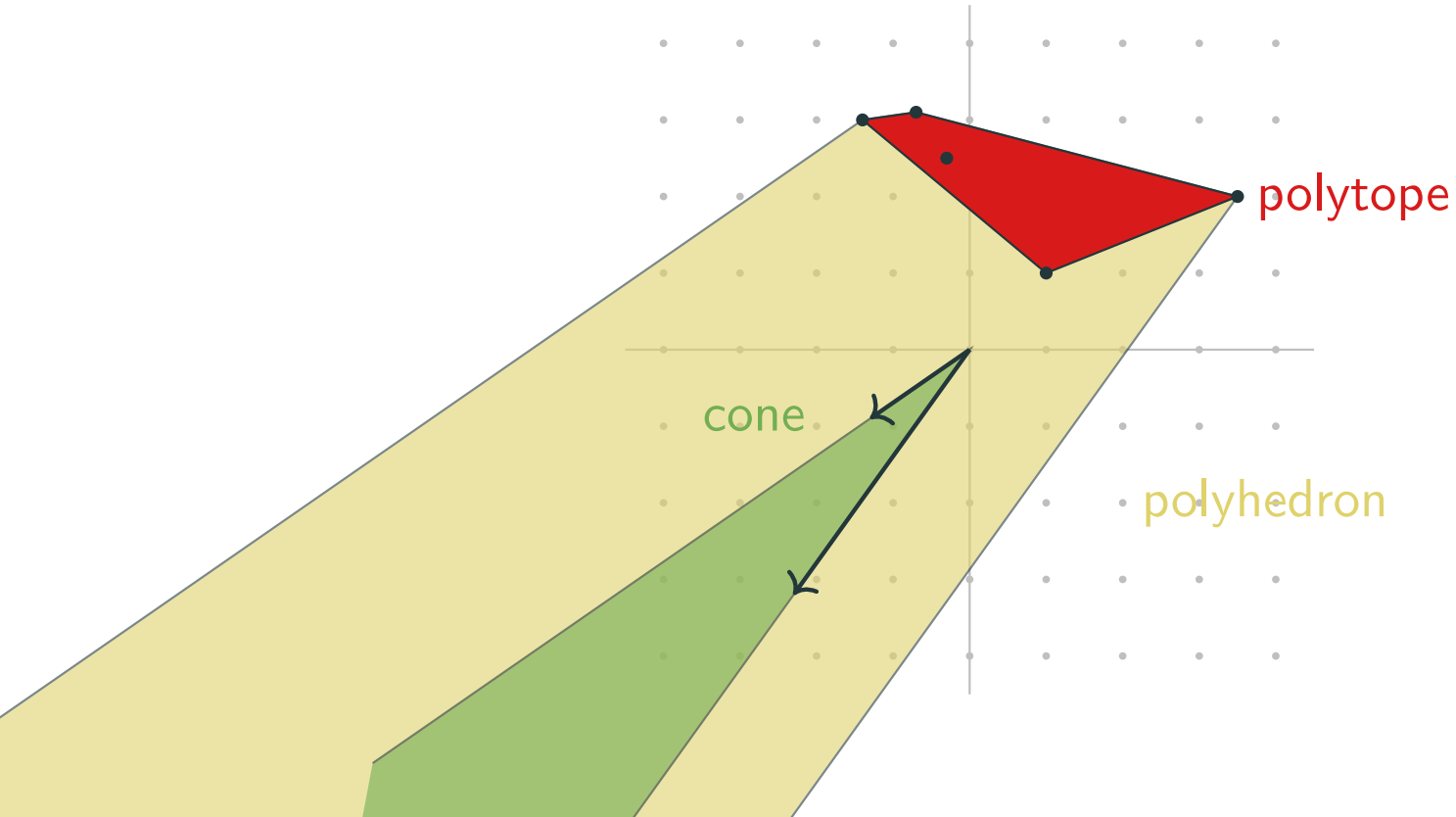
- polyhedron: polytope + finitely generated cone



Roadmap

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$$(hull(X) + cone(V)) \cap \mathbb{Z}^n = \varnothing \iff (hull(X) + C) \cap \mathbb{Z}^n = \varnothing \qquad \textit{by Thm}$$

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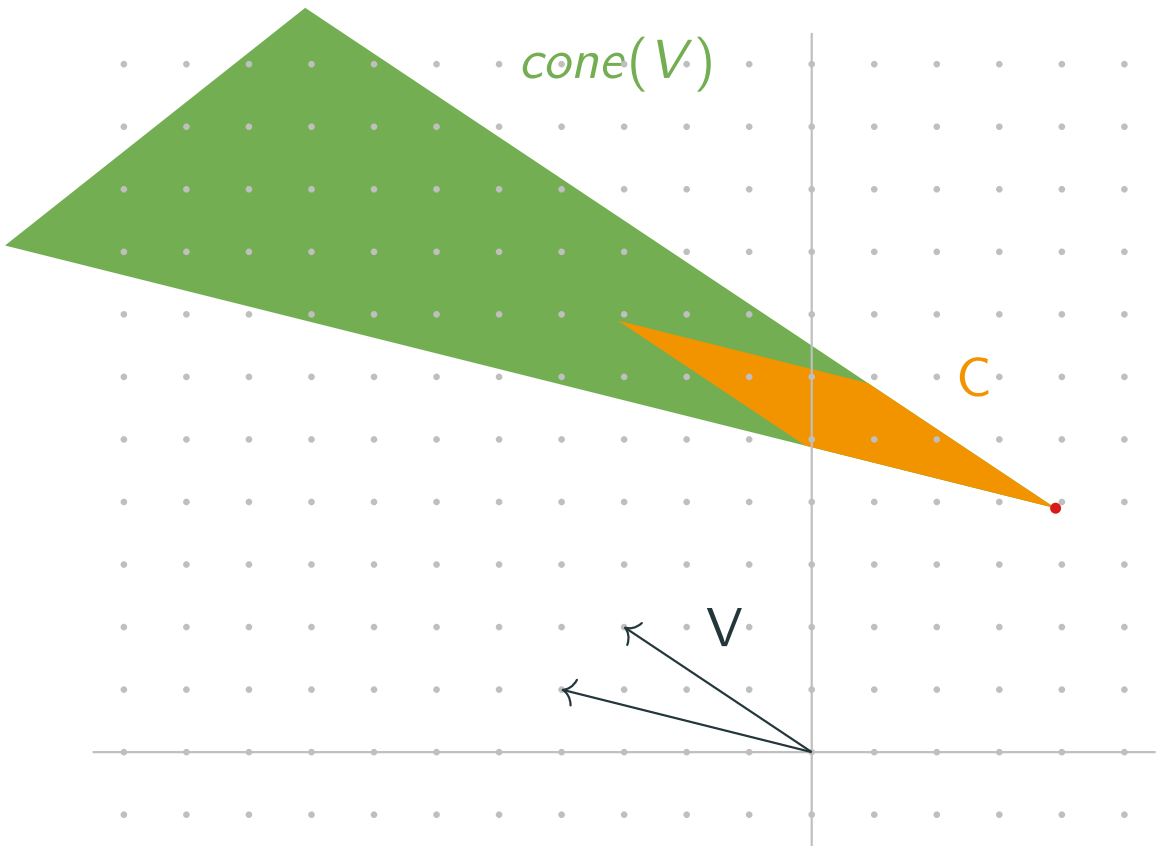
$(Y + cone(V)) \cap \mathbb{Z}^n = \varnothing \Longleftarrow (Y + C) \cap \mathbb{Z}^n = \varnothing$ *for Y convex*

Proof (by picture).



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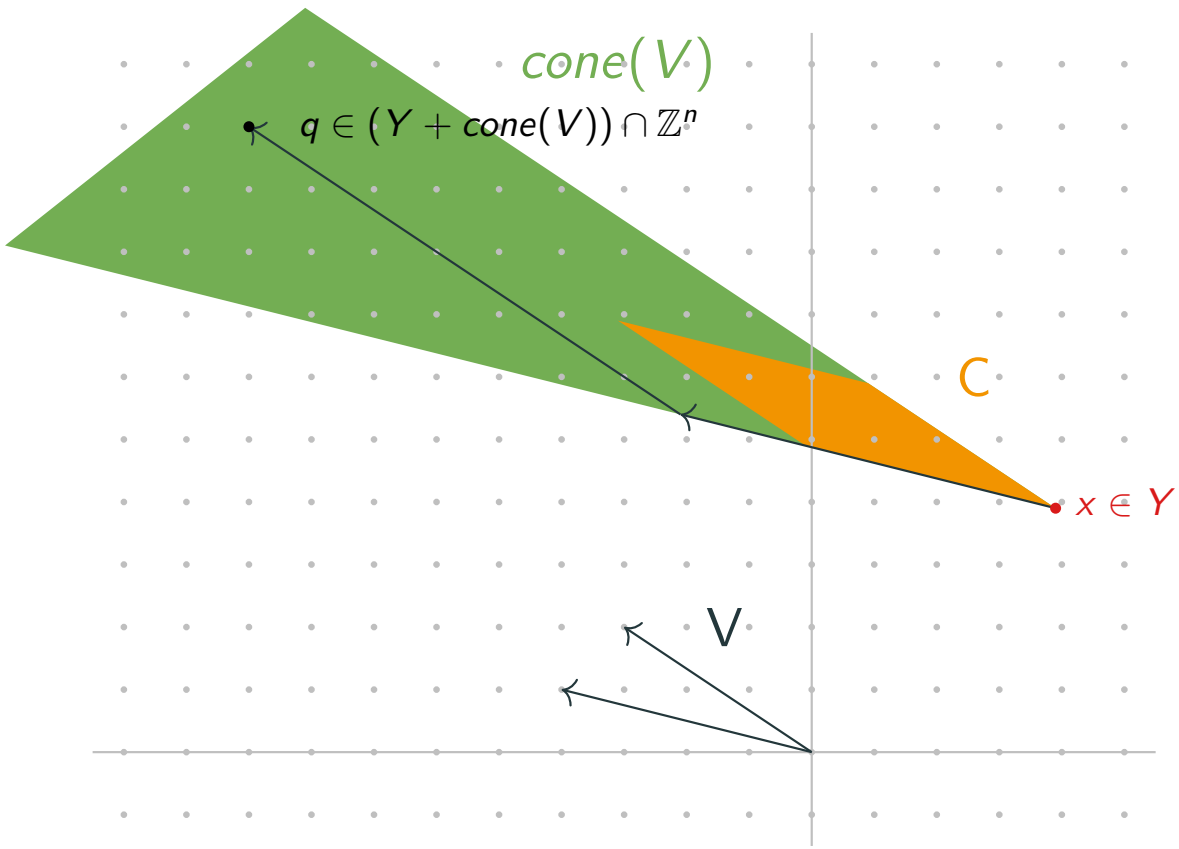

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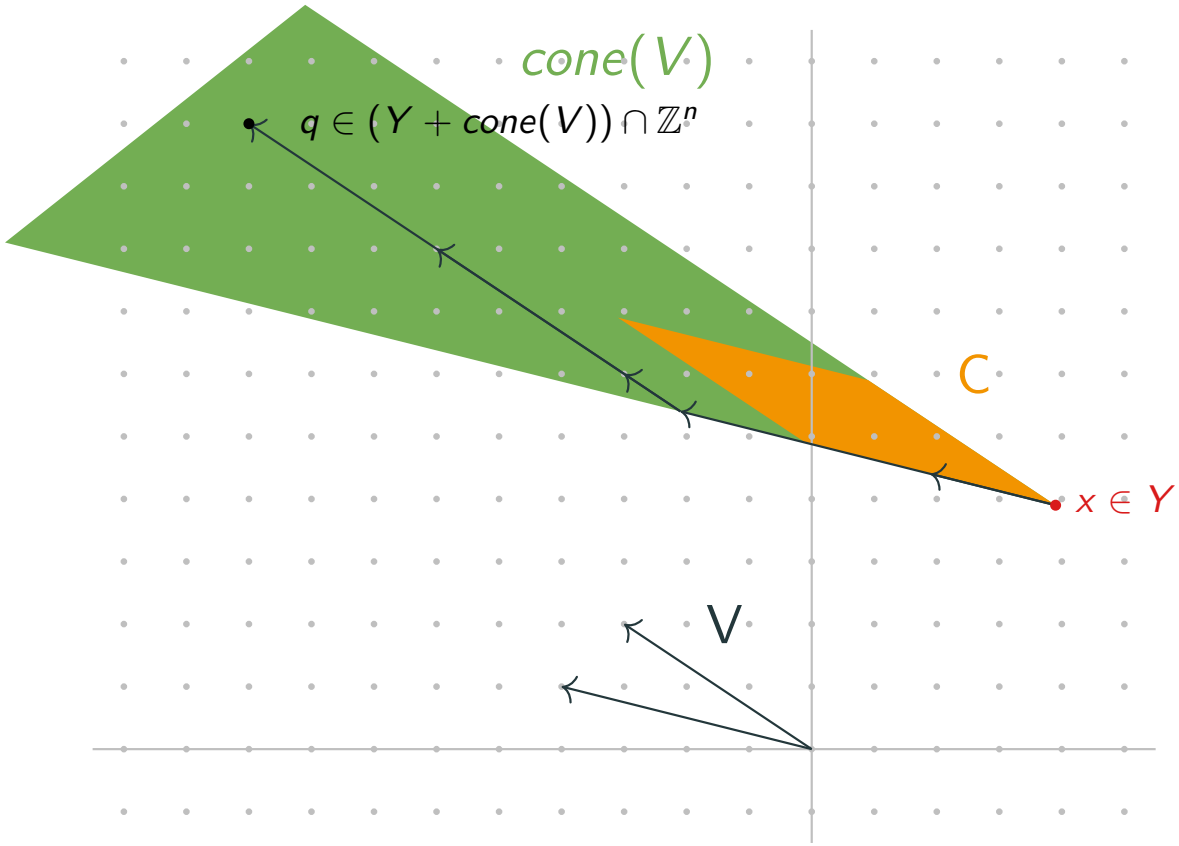

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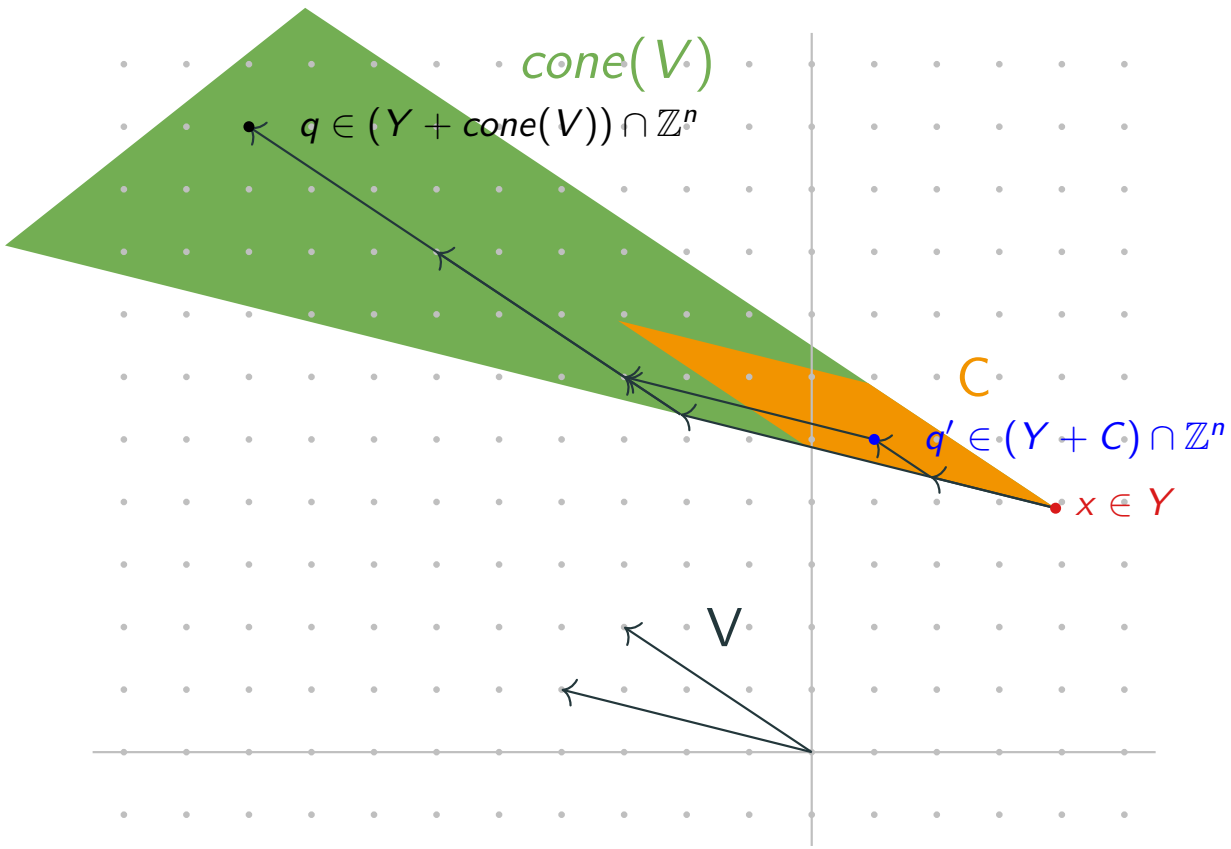

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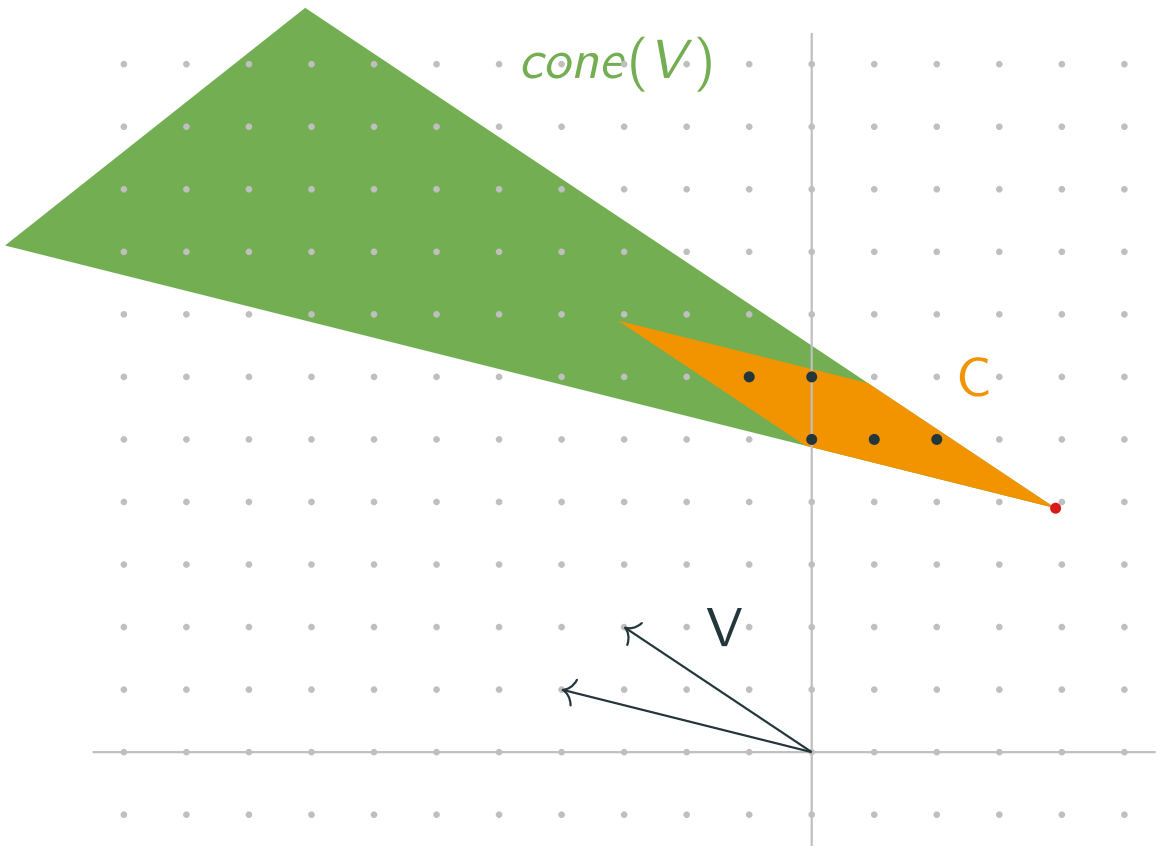

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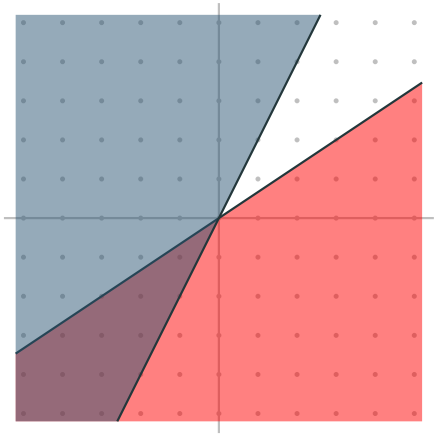

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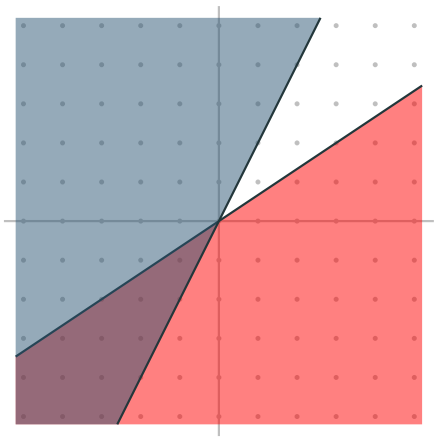

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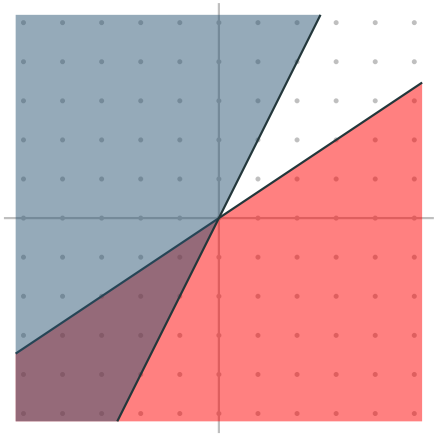

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i.e. $\exists v_1, \ldots, v_m$ such that $C = cone(v_1, \ldots, v_m)$

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Aim

convert $\{\overline{x} \mid A\overline{x} \leqslant \overline{b}\}$ into $hull(X) + cone(V)$

Construction

- define polyhedral cone C

$$C = \left\{ \begin{pmatrix} \overline{x} \\ \tau \end{pmatrix} \middle| \tau \geqslant 0, A\overline{x} - \tau\overline{b} \leqslant \overline{0} \right\} = \left\{ \overline{y} \middle| \begin{pmatrix} A & -\overline{b} \\ \overline{0} & -1 \end{pmatrix} \overline{y} \leqslant \overline{0} \right\}$$

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for $\mathbb{Q}^3$ can take cross-product

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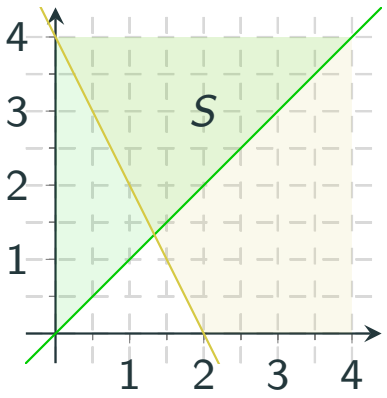
■

Example

- consider $x \leqslant y$ and $4 - 2x \leqslant y$



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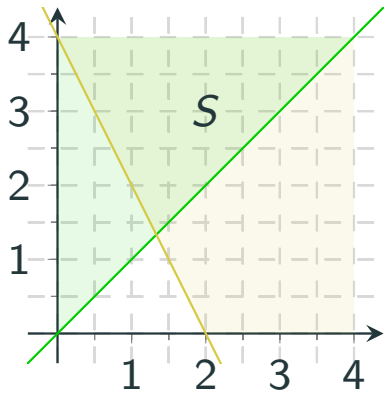

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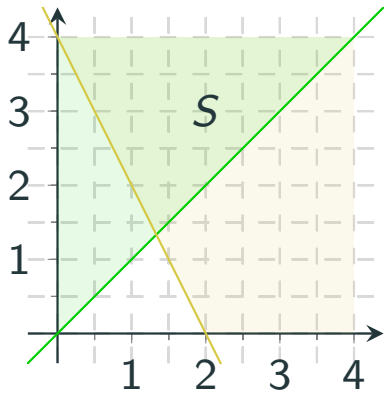

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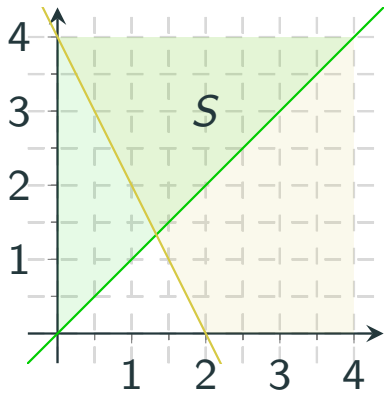

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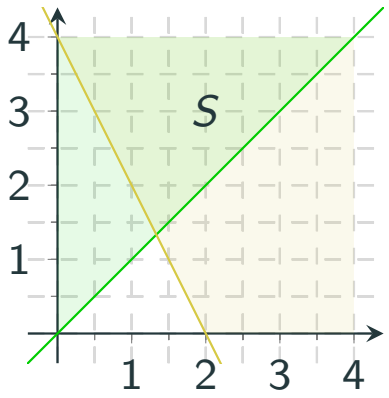

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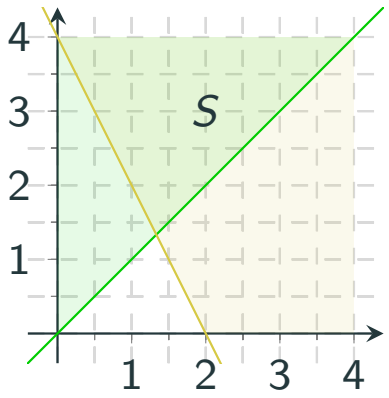

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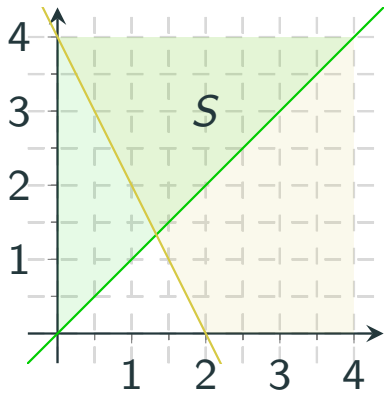

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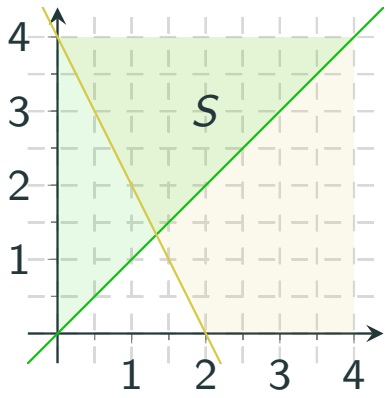

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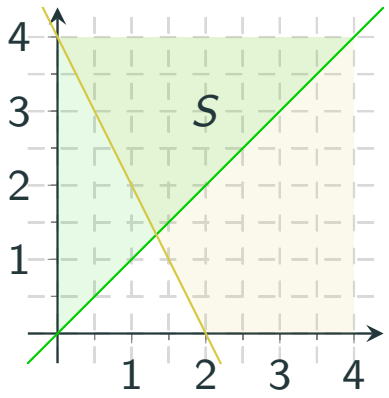

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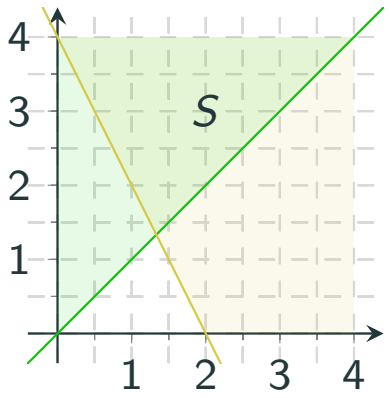

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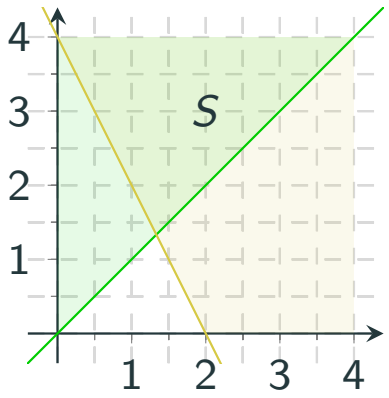

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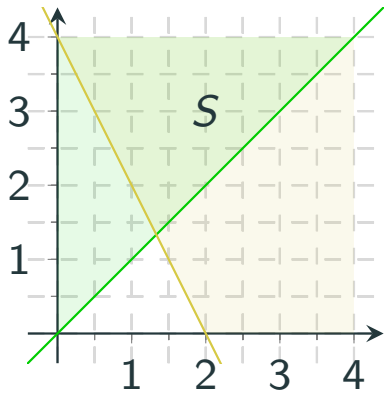

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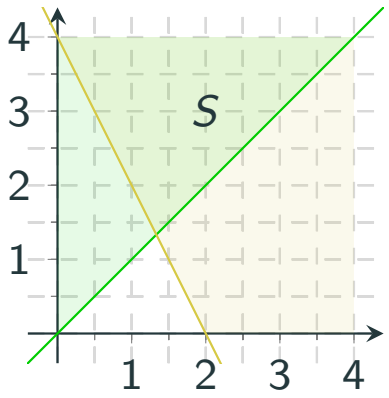

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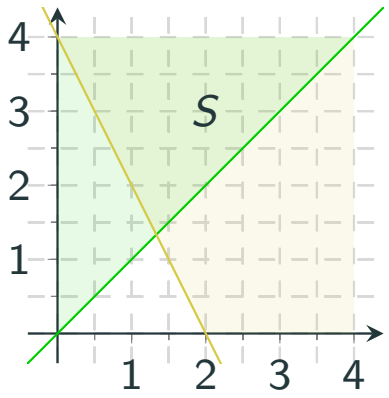

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    $c_{23} \cdot w_1 = 3 \qquad c_{23} \cdot w_2 = 0 \qquad c_{23} \cdot w_3 = 0$

- for $A' = \begin{pmatrix} 4 & 4 & 3 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix}$ have $cone\,(W) = \{\overline{x} \mid A'\overline{x} \leqslant 0\}$
- $\{\overline{x} \mid A\overline{x} \leqslant 0\} = cone\,(\{v_1, v_2, v_3\}) = cone\,(\{(\frac{4}{3} \quad \frac{4}{3} \quad 1)^T, (1 \quad 1 \quad 0)^T, (-1 \quad 2 \quad 0)^T\})$
- $S = hull\,(\frac{4}{3} \quad \frac{4}{3})^T + cone\,\{(1 \quad 1)^T, (-1 \quad 2)^T\}$
- $S \cap \mathbb{Z}$ has bound $B := h \cdot (1 + n) = 2 \cdot 3 = 6$, where h is maximal coefficient in cone+hull