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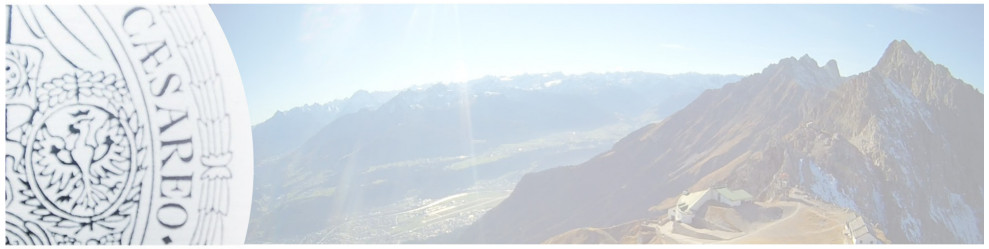

# SAT and SMT Solving

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## Outline

- Summary of Last Week
- Cutting Planes
- Bounds for Integer Solutions



### Idea (Branch and Bound)

- given $\mathbb{Q}^2$ solution $\alpha$, add constraints to exclude $\alpha$ but preserve $\mathbb{Z}^2$ solutions: if $a < \alpha(x) < a_1$, use Simplex on problems $C \wedge x \leqslant a$ and $C \wedge x \geqslant a + 1$
- need not terminate if solution space is unbounded

**Algorithm** BranchAndBound($\varphi$)

**Input:** LIA constraint $\varphi$
**Output:** unsatisfiable, or satisfying assignment

```
let res be result of deciding φ over ℚ          ▷ e.g. by Simplex
if res is unsatisfiable then
    return unsatisfiable
else if res is solution over ℤ then
    return res
else
    let x be variable assigned non-integer value q in res
    res = BranchAndBound(φ ∧ x ⩽ ⌊q⌋)
    return res ≠ unsatisfiable ? res : BranchAndBound(φ ∧ x ⩾ ⌈q⌉)
```



### Definition

$\mathbb{Q}^2$-solution space of linear arithmetic problem $Ax \leqslant b$ is bounded if for all $x_i$ there exist $l_i, u_i \in \mathbb{Q}$ such that all $\mathbb{Q}^2$-solutions $v$ satisfy $l_i \leqslant v(x_i) \leqslant u_i$

### Theorem

*If solution space to $\varphi$ is bounded then BranchAndBound($\varphi$) returns unsatisfiable iff $\varphi$ has no solution in $\mathbb{Z}^2$*

## Fourier-Motzkin Elimination

### Aim

build theory solver for linear rational arithmetic (LRA):
decide whether conjunction of linear (in)equalities $\varphi$ is satisfiable over $\mathbb{Q}$

### Preprocessing: eliminate $\neq$

$(t_1 \neq t_2) \wedge \varphi$ is satisfiable iff $(t_1 < t_2) \wedge \varphi$ or $(t_1 > t_2) \wedge \varphi$ are satisfiable

### Definition (Elimination step)

- for variable $x$ in $\varphi$, can write $\varphi$ as

$$\bigwedge_i (x < U_i) \;\wedge\; \bigwedge_j (x \leqslant u_j) \;\wedge\; \bigwedge_k (L_k < x) \;\wedge\; \bigwedge_m (\ell_m \leqslant x) \;\wedge\; \psi$$

  where $U_i$, $u_j$, $L_k$, $\ell_m$, $\psi$ are without $x$

- let $elim(\varphi, x)$ be conjunction of

$$\bigwedge_i \bigwedge_k (L_k < U_i) \quad \bigwedge_i \bigwedge_m (\ell_m < U_i) \quad \bigwedge_j \bigwedge_k (L_k < u_j) \quad \bigwedge_j \bigwedge_m (\ell_m \leqslant u_j) \quad \psi$$

### Lemma

*$\varphi$ is LRA-satisfiable iff $elim(\varphi, x)$ is LRA-satisfiable*



### Observation

- can subsequently eliminate all variables
- checking satisfiability of formula without variables is easy
- so obtain decision procedure for LRA!

### Example (Fourier-Motzkin elimination)

$$\begin{aligned} 2x - 4y &\leqslant 8 \\ x + y + z &> 3 \\ 3y + 2z &< 5 \\ y - z &\geqslant 0 \end{aligned} \quad \text{i.e.} \quad \begin{aligned} x &\leqslant 4 + 2y \\ x &> 3 - y - z \end{aligned} \quad \xRightarrow[\text{eliminate } x]{}$$

$$\begin{aligned} 3 - y - z &< 4 + 2y \\ 3y + 2z &< 5 \\ y - z &\geqslant 0 \end{aligned} \quad \text{i.e.} \quad \begin{aligned} y &> -\tfrac{1}{3}z - \tfrac{1}{3} \\ y &< \tfrac{5}{3} - \tfrac{2}{3}z \\ y &\geqslant z \end{aligned} \quad \xRightarrow[\text{eliminate } y]{}$$

$$\begin{aligned} -\tfrac{1}{3}z - \tfrac{1}{3} &< \tfrac{5}{3} - \tfrac{2}{3}z \\ z &< \tfrac{5}{3} - \tfrac{2}{3}z \end{aligned} \quad \text{i.e.} \quad \begin{aligned} z &< 6 \\ z &< 1 \end{aligned} \quad \xRightarrow[\text{eliminate } z]{}$$

(empty constraints) satisfiable

### Remark

worst-case complexity of FME is double exponential in number of variables



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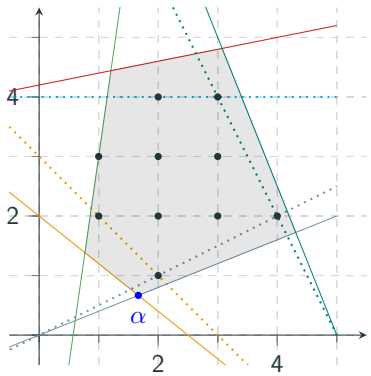
Consider set of constraints over linear integer arithmetic.

### Example




### Definition (Cut)

given solution $\alpha$ over $\mathbb{Q}^n$, cut is inequality $a_1x_1 + \cdots + a_nx_n \leqslant b$
which is not satisfied by $\alpha$ but by every $\mathbb{Z}^n$-solution

need not terminate
for unbounded problems

### Solving Strategy

like in BranchAndBound, keep adding cuts until integer solution found

## Gomory Cuts: Assumptions

- Simplex returned solution $\alpha$ over $\mathbb{Q}^n$:
  final tableau is $A$ with dependent variables $D$ and independent variables $I$

$$A\overline{x}_I = \overline{x}_D \tag{1}$$
$$l_k \leqslant x_k \leqslant u_k \quad \forall x_k \tag{2}$$

- for some $x_i \in D$ its value $\alpha(x_i) \notin \mathbb{Z}$
- for all $x_j \in I$ value $\alpha(x_j)$ is $l_j$ or $u_j$ (by definition of Simplex)

## Notation

- write $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$
- by assumption all independent variables are assigned bounds, so can split

$$L = \{ x_j \in I \mid \alpha(x_j) = l_j \} \qquad U = \{ x_j \in I \mid \alpha(x_j) = u_j \}$$
$$L^+ = \{ x_j \in L \mid A_{ij} \geqslant 0 \} \qquad U^+ = \{ x_j \in U \mid A_{ij} \geqslant 0 \}$$
$$L^- = \{ x_j \in L \mid A_{ij} < 0 \} \qquad U^- = \{ x_j \in U \mid A_{ij} < 0 \}$$

## Lemma (Gomory Cut)

*the following inequality is a cut:*

$$\sum_{x_j \in L^+} \frac{A_{ij}}{1-c}(x_j - l_j) - \sum_{x_j \in U^-} \frac{A_{ij}}{1-c}(u_j - x_j) - \sum_{x_j \in L^-} \frac{A_{ij}}{c}(x_j - l_j) + \sum_{x_j \in U^+} \frac{A_{ij}}{c}(u_j - x_j) \geqslant 1$$



$$A\overline{x}_I = \overline{x}_D \tag{1}$$
$$l_k \leqslant x_k \leqslant u_k \quad \forall x_k \tag{2}$$

## Proof (1)

- set up conditions for integer solution $\overline{x}$ to (1) and (2)
- $\overline{x}$ satisfies $i$-th row of (1):

$$x_i = \sum_{x_j \in I} A_{ij} x_j \tag{3}$$

- because $\alpha$ is solution, it holds that

$$\alpha(x_i) = \sum_{x_j \in I} A_{ij} \alpha(x_j) \tag{4}$$

- subtract (4) from (3):

$$\begin{aligned} x_i - \alpha(x_i) &= \sum_{x_j \in I} A_{ij}(x_j - \alpha(x_j)) \\ &= \sum_{x_j \in L} A_{ij}(x_j - l_j) - \sum_{x_j \in U} A_{ij}(u_j - x_j) \end{aligned} \tag{5}$$



## Proof (2)

- have

$$x_i - \alpha(x_i) = \underbrace{\sum_{x_j \in L} A_{ij}(x_j - l_j)}_{\mathcal{L}} - \underbrace{\sum_{x_j \in U} A_{ij}(u_j - x_j)}_{\mathcal{U}} \tag{5}$$

- for $c = \alpha(x_i) - \lfloor \alpha(x_i) \rfloor$ have $0 < c < 1$, can write $\alpha(x_i) = \lfloor \alpha(x_i) \rfloor + c$, so

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- for integer solution $\overline{x}$ left-hand side must be integer, so also right-hand side
- abbreviate

$$\mathcal{L}^+ = \sum_{x_j \in L^+} A_{ij}(x_j - l_j) \qquad \mathcal{U}^+ = \sum_{x_j \in U^+} A_{ij}(u_j - x_j)$$
$$\mathcal{L}^- = \sum_{x_j \in L^-} A_{ij}(x_j - l_j) \qquad \mathcal{U}^- = \sum_{x_j \in U^-} A_{ij}(u_j - x_j)$$

  so $\mathcal{L} = \mathcal{L}^+ + \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ + \mathcal{U}^-$
- have $\mathcal{L}^+ \geqslant 0$, $\mathcal{U}^+ \geqslant 0$ and $\mathcal{L}^- \leqslant 0$, $\mathcal{U}^- \leqslant 0$
- distinguish $\mathcal{L} \geqslant \mathcal{U}$ or $\mathcal{L} < \mathcal{U}$



## Proof (3)

- both sides are integer in equation

$$x_i - \lfloor \alpha(x_i) \rfloor = c + \mathcal{L} - \mathcal{U} \tag{6}$$

- if $\mathcal{L} \geqslant \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \geqslant 1$ because integer, so $\mathcal{L} - \mathcal{U} \geqslant 1 - c$
  - in particular $\mathcal{L}^+ - \mathcal{U}^- \geqslant 1 - c$
  - $$\frac{1}{1-c}(\mathcal{L}^+ - \mathcal{U}^-) \geqslant 1 \tag{7}$$

since $\mathcal{L}^+ \geqslant \mathcal{L}$ and $\mathcal{U}^- \leqslant \mathcal{U}$

- otherwise $\mathcal{L} < \mathcal{U}$:
  - have $c + \mathcal{L} - \mathcal{U} \leqslant 0$ because integer, so $\mathcal{U} - \mathcal{L} \geqslant c$
  - in particular $\mathcal{U}^+ - \mathcal{L}^- \geqslant c$
  - $$\frac{1}{c}(\mathcal{U}^+ - \mathcal{L}^-) \geqslant 1 \tag{8}$$

since $\mathcal{U}^+ \geqslant \mathcal{U}$ and $\mathcal{L}^- \leqslant \mathcal{L}$

- terms $\mathcal{L}^+$, $\mathcal{U}^+$, $-\mathcal{L}^-$ and $-\mathcal{U}^-$ always non-negative, as
- add (7) and (8) to obtain cut

$$\frac{1}{1-c}(\mathcal{L}^+ - \mathcal{U}^-) + \frac{1}{c}(\mathcal{U}^+ - \mathcal{L}^-) \geqslant 1$$

the desired monster inequality!

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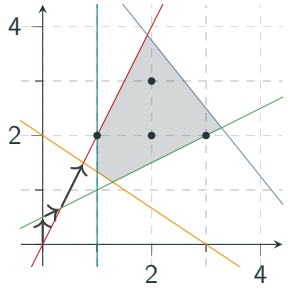
## Example




$$-2x - 3y \leqslant -6$$
$$-2x + y \leqslant 0$$
$$x - 2y \leqslant -1$$
$$5x + 4y \leqslant 25$$

- infinite $\mathbb{Q}^2$-solution space
- four solutions in $\mathbb{Z}^2$
- Simplex solution search

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} -2 & -3 \\ -2 & 1 \\ 1 & -2 \\ 5 & 4 \end{pmatrix} \end{array} \quad \begin{array}{l} s_1 \leqslant -6 \\ s_2 \leqslant \phantom{-}0 \\ s_3 \leqslant -1 \\ s_4 \leqslant \phantom{-}25 \end{array} \quad \longrightarrow \quad \begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \\ -\frac{7}{8} & -\frac{13}{8} \end{pmatrix} \end{array} \quad \begin{array}{ll} x = \frac{3}{4} & s_1 = -6 \\ y = \frac{3}{2} & s_2 = \phantom{-}0 \\ & s_3 = -2\frac{1}{4} \\ & s_4 = \phantom{-}9\frac{3}{4} \end{array}$$

initial tableau — final tableau — solution

- independent variables $s_2 = 0$ and $s_1 = -6$ at bounds, basic $x$ is assigned $\frac{3}{4} \notin \mathbb{Z}$
- from $c = \frac{3}{4}$ obtain Gomory cut $4(\frac{3}{8}(0 - s_2) + \frac{1}{8}(-6 - s_1)) \geqslant 1$
- corresponds to $-\frac{3}{2}(-2x + y) - \frac{1}{2}(-2x - 3y) \geqslant 4$, simplified $x \geqslant 1$

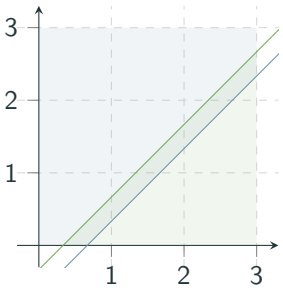


## Outline





## Example




- $3x - 3y \geqslant 1 \wedge 3x - 3y \leqslant 2$
- unbounded problem
- no solution in $\mathbb{Z}^2$
- BranchAndBound adding (Gomory) cuts need not terminate

## Good News

- given (potentially unbounded) linear arithmetic problem $A\overline{x} \leqslant \overline{b}$
- one can compute bound $B$ from $A$ and $\overline{b}$ such that

$$\exists \overline{x} \in \mathbb{Z}^n \text{ with } A\overline{x} \leqslant \overline{b} \quad \Longrightarrow \quad \overline{x} \in \{-B, \ldots, B\}^n$$

- obtain equisatisfiable bounded problem by adding $-B \leqslant x_i \leqslant B$

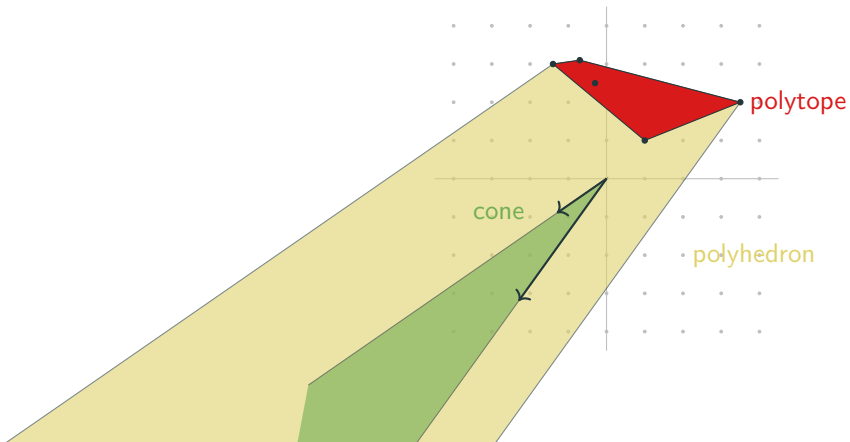
(material in the remainder of this section is by René Thiemann)



## Geometric Objects

### Definitions

- polytope: convex hull of finite set of vectors $X$
  smallest $V \supseteq X$ s.t. $\forall v, w \in V$, $0 \leqslant \lambda \leqslant 1$ have $v\lambda + (1 - \lambda)w \in V$
- cone: non-negative linear combinations of finite set of vectors $V$
- polyhedron: polytope + finitely generated cone



## Roadmap

1. represent $\{\overline{x} \mid A\overline{x} \leqslant \overline{b}\}$ as $hull(X) + cone(V)$
   - using representation of $\{\overline{x} \mid A\overline{x} \leqslant \overline{0}\}$ as $cone(V)$
   - construction of generators in FMW theorem
2. derive bound $B$ for hull + cone representation:

$$(hull(X) + cone(V)) \cap \mathbb{Z}^n = \varnothing$$
$$\iff$$
$$(hull(X) + cone(V)) \cap \{-B, \dots, B\}^n = \varnothing$$



## Integer Solutions of Polyhedra

Consider bounded set $X \subseteq \mathbb{Q}^n$ and $V \subseteq \mathbb{Z}^n$ such that $V = \{v_1, \dots, v_n\}$

**Notation**

$C = \{\sum_{i=1}^n \lambda_i \cdot v_i \mid v_i \in V \wedge 0 \leqslant \lambda_i \leqslant 1\}$

yet to be proven ...

**Theorem**

$(Y + cone(V)) \cap \mathbb{Z}^n = \varnothing \iff (Y + C) \cap \mathbb{Z}^n = \varnothing$ *(if $Y$ convex)*

**Observation**

- have $C \subseteq cone(V)$ by definition, so $(X + C) \subseteq (X + cone(V))$
- so direction $\Longrightarrow$ is easy

**Corollary**

*Suppose $|c| \leqslant b$ for all coefficients $c$ of vectors in $X \cup V$.*
*For $B := b \cdot (1 + n)$ have*

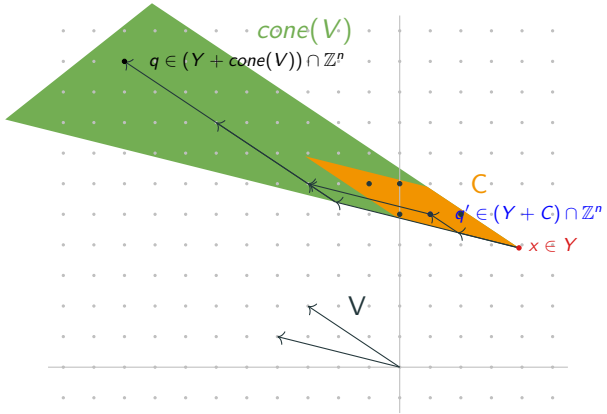
$$(hull(X) + cone(V)) \cap \mathbb{Z}^n = \varnothing \iff (hull(X) + C) \cap \mathbb{Z}^n = \varnothing \quad \text{by Thm}$$
$$\iff (hull(X) + C) \cap \{-B, \dots, B\}^n = \varnothing$$



**Theorem**

$(Y + cone(V)) \cap \mathbb{Z}^n = \varnothing \Longleftarrow (Y + C) \cap \mathbb{Z}^n = \varnothing$ *for $Y$ convex*

**Proof (by picture).**

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## Roadmap

1. represent $\{\overline{x} \mid A\overline{x} \leqslant \overline{b}\}$ as $hull(X) + cone(V)$
   - using representation of $\{\overline{x} \mid A\overline{x} \leqslant \overline{0}\}$ as $cone(V)$
   - construction of generators in FMW theorem
2. derive bound $B$ for hull + cone representation: ✓

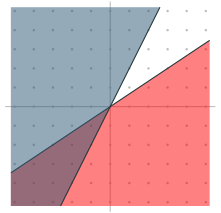
$$(hull(X) + cone(V)) \cap \mathbb{Z}^n = \varnothing$$
$$\iff$$
$$(hull(X) + cone(V)) \cap \{-B, \dots, B\}^n = \varnothing$$

## Polyhedral Cones

**Definition**
set of vectors $C$ is polyhedral cone if $C = \{\overline{x} \mid A\overline{x} \leqslant \overline{0}\}$ for some matrix $A$

**Lemma**
*$C$ is polyhedral cone iff $C$ is intersection of finitely many half-spaces*

**Example**



$$A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$$

$$2x - y \leqslant 0 \iff y \geqslant 2x$$

$$-2x + 3y \leqslant 0 \iff y \leqslant \frac{2}{3}x$$

i.e. $\exists v_1, \ldots, v_m$ such that $C = cone(v_1, \ldots, v_m)$

**Theorem (Farkas, Minkowski, Weyl)**
*A cone $C$ is polyhedral iff it is finitely generated*



**Aim**
convert $\{\overline{x} \mid A\overline{x} \leqslant \overline{b}\}$ into $hull(X) + cone(V)$

**Construction**

- define polyhedral cone $C$

$$C = \left\{ \begin{pmatrix} \overline{x} \\ \tau \end{pmatrix} \middle| \tau \geqslant 0, A\overline{x} - \tau\overline{b} \leqslant \overline{0} \right\} = \left\{ \overline{y} \middle| \begin{pmatrix} A & -\overline{b} \\ \overline{0} & -1 \end{pmatrix} \overline{y} \leqslant \overline{0} \right\}$$

- using FMW theorem $\exists$ finite set of vectors such that

$$C = cone \left\{ \begin{pmatrix} x_1 \\ \tau_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_\ell \\ \tau_\ell \end{pmatrix}, \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_k \\ 0 \end{pmatrix} \right\}$$

  where all $\tau_i > 0$, so can define $\overline{y}_i = \frac{1}{\tau_i}\overline{x}_i$ define $\overline{z}_j = |\prod \text{denominators of } \overline{u}_j| \cdot \overline{u}_j$, so $z_j$ is integral

**Claim**
$\{\overline{x} \mid A\overline{x} \leqslant \overline{b}\} = hull\,\{\overline{y}_1, \ldots, \overline{y}_\ell\} + cone\,\{\overline{z}_1, \ldots, \overline{z}_k\}$



**Claim**
$\{\overline{x} \mid A\overline{x} \leqslant \overline{b}\} = hull\,\{\overline{y}_1, \ldots, \overline{y}_\ell\} + cone\,\{\overline{z}_1, \ldots, \overline{z}_k\}$

**Proof.**

$$C = \left\{ \begin{pmatrix} \overline{x} \\ \tau \end{pmatrix} \middle| \tau \geqslant 0, A\overline{x} - \tau\overline{b} \leqslant \overline{0} \right\} = cone \left\{ \begin{pmatrix} \overline{y}_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} \overline{z}_1 \\ 0 \end{pmatrix}, \ldots \right\}$$

$$\begin{aligned} A\overline{x} \leqslant \overline{b} &\iff \begin{pmatrix} \overline{x} \\ 1 \end{pmatrix} \in C \\ &\iff \begin{pmatrix} \overline{x} \\ 1 \end{pmatrix} = \sum \lambda_i \begin{pmatrix} \overline{y}_i \\ 1 \end{pmatrix} + \sum \kappa_j \begin{pmatrix} \overline{z}_j \\ 0 \end{pmatrix} \text{ with } \lambda_1, \ldots, \kappa_1, \ldots \geqslant 0 \\ &\iff \overline{x} = (\sum \lambda_i \overline{y}_i) + (\sum \kappa_j \overline{z}_j) \text{ and } \sum \lambda_i = 1 \\ &\iff \overline{x} = \overline{y} + \overline{z} \text{ with } \overline{y} \in hull\,\{\overline{y}_1, \ldots\}, \overline{z} \in cone\,\{\overline{z}_1, \ldots\} \end{aligned}$$

■



## Roadmap

1. represent $\{\overline{x} \mid A\overline{x} \leqslant \overline{b}\}$ as $hull(X) + cone(V)$ ✓
   - using representation of $\{\overline{x} \mid A\overline{x} \leqslant \overline{0}\}$ as $cone(V)$
   - construction of generators in FMW theorem details
2. derive bound $B$ for hull + cone representation: ✓

$$\begin{aligned} &(hull(X) + cone(V)) \cap \mathbb{Z}^n = \varnothing \\ &\iff \\ &(hull(X) + cone(V)) \cap \{-B, \ldots, B\}^n = \varnothing \end{aligned}$$

**Bottom line**
for every LIA problem can compute bounds to get equisatisfiable bounded problem, so BranchAndBound terminates

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## Bounds for FMW Theorem

**Theorem (Farkas, Minkowski, Weyl)**
*A cone is polyhedral iff it is finitely generated.*

**Proof (construction)**
$\Longleftarrow$: finitely generated implies polyhedral

for $\mathbb{Q}^3$ can take cross-product

- consider $cone\,(V)$ for $V = \{\overline{v}_1, \ldots, \overline{v}_m\} \subseteq \mathbb{Q}^n$
- for every set $W = \{\overline{w}_1, \ldots, \overline{w}_{n-1}\} \subseteq V$ of linearly independent vectors: compute vector $\overline{c}_W$ normal to hyper-space spanned by $W$
  - if $\overline{v}_i \cdot \overline{c}_W \leqslant 0$ for all $i$, then add $\overline{c}_W$ as row to $A$
  - if $\overline{v}_i \cdot \overline{c}_W \geqslant 0$ for all $i$, then add $-\overline{c}_W$ as row to $A$
- $cone\,(V) = \{\overline{x} \mid A\overline{x} \leqslant \overline{0}\}$

back



**Theorem (Farkas, Minkowski, Weyl)**
*A cone is polyhedral iff it is finitely generated.*

**Proof (construction).**
$\Longrightarrow$: polyhedral implies finitely generated

- consider $\{\overline{x} \mid A\overline{x} \leqslant \overline{0}\}$
- define $W$ as the set of row vectors of $A$
- by first direction obtain $A'$ such that $cone\,(W) = \{\overline{x} \mid A'\overline{x} \leqslant \overline{0}\}$
- define $V$ as the set of row vectors of $A'$
- $\{\overline{x} \mid A\overline{x} \leqslant \overline{0}\} = cone\,(V)$

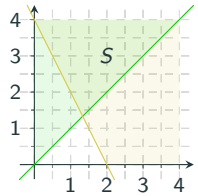
■



## Example

- consider $x \leqslant y$ and $4 - 2x \leqslant y$

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ -2 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}}_{A} \cdot \begin{pmatrix} x \\ y \\ \tau \end{pmatrix} \leqslant 0$$




- use proof of FMW theorem: compute $cone\,(W)$ for $W = \{w_1, w_2, w_3\}$
  $w_1 = (1 \quad -1 \quad 0)^T \qquad w_2 = (-2 \quad -1 \quad 4)^T \qquad w_3 = (0 \quad 0 \quad -1)^T$
  - $c_{12} = w_1 \times w_2 = (-4 \quad -4 \quad -3)$ is normal to $w_1$ and $w_2$
    $c_{12} \cdot w_1 = 0 \qquad c_{12} \cdot w_2 = 0 \qquad c_{12} \cdot w_3 = 3$
  - $c_{13} = w_1 \times w_3 = (1 \quad 1 \quad 0)$ is normal to $w_1$ and $w_3$
    $c_{13} \cdot w_1 = 0 \qquad c_{13} \cdot w_2 = -3 \qquad c_{13} \cdot w_3 = 0$
  - $c_{23} = w_2 \times w_3 = (1 \quad -2 \quad 0)$ is normal to $w_2$ and $w_3$
    $c_{23} \cdot w_1 = 3 \qquad c_{23} \cdot w_2 = 0 \qquad c_{23} \cdot w_3 = 0$
- for $A' = \begin{pmatrix} 4 & 4 & 3 \\ 1 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix}$ have $cone\,(W) = \{\overline{x} \mid A'\overline{x} \leqslant 0\}$
- $\{\overline{x} \mid A\overline{x} \leqslant 0\} = cone\,(\{v_1, v_2, v_3\}) = cone\,(\{(\frac{4}{3} \quad \frac{4}{3} \quad 1)^T, (1 \quad 1 \quad 0)^T, (-1 \quad 2 \quad 0)^T\})$
- $S = hull\,(\frac{4}{3} \quad \frac{4}{3})^T + cone\,\{(1 \quad 1)^T, (-1 \quad 2)^T\}$
- $S \cap \mathbb{Z}$ has bound $B := b \cdot (1 + n) = 2 \cdot 3 = 6$, where $b$ is maximal coefficient in cone+hull