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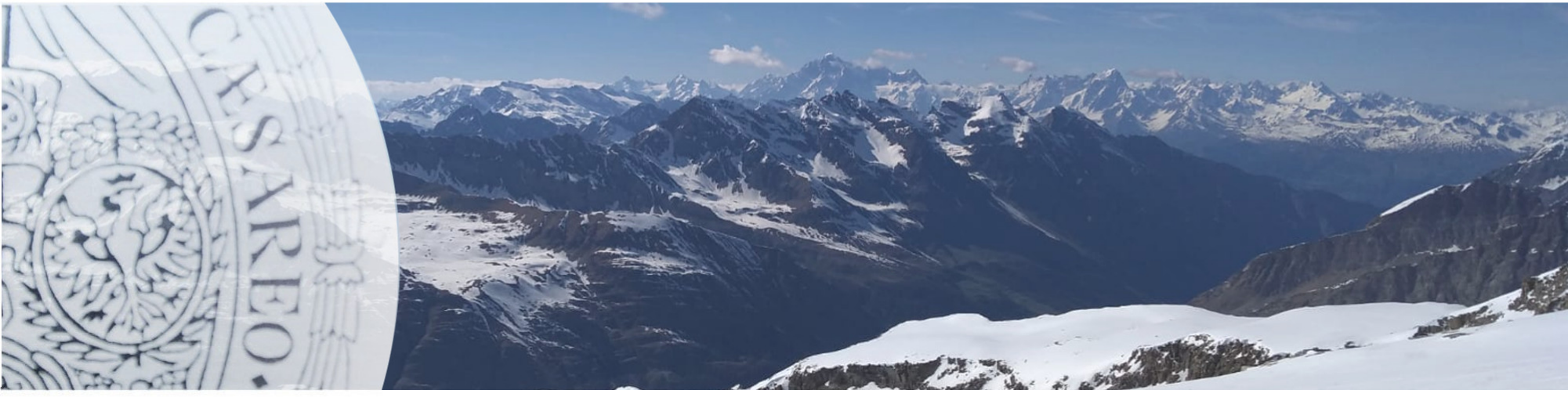

SAT and SMT Solving

Sarah Winkler

KRDB
Department of Computer Science
Free University of Bozen-Bolzano

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WS 2022

Outline

- Summary of Last Week
- Nelson-Oppen Combination Method
- Application: Collision Attacks

Definition (Bit Vector Theory)

- variable $\mathbf{x}_k$ is list of length k of propositional variables $x_{k-1} \ldots x_2 x_1 x_0$
- constant n_k is bit list of length k
- formulas built according to grammar

$$
\begin{aligned}
\mathit{formula} &:= (\mathit{formula} \vee \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\neg \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &:= \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{true} \mid \mathit{false} \\
\mathit{rel} &:= \ = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &:= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{var} \mid \mathit{constant} \mid \mathit{term}[i{:}j] \mid \\
&\qquad (\mathit{formula}\ ?\ \mathit{term} : \mathit{term}) \\
\mathit{binop} &:= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid \mid \mid \hat{} \mid :: \\
\mathit{unop} &:= \sim \mid -
\end{aligned}
$$

- axioms are equality axioms plus rules for arithmetic/comparison/bitwise operations on bit vectors of length k
- solution assigns bit list of length k to variables $\mathbf{x}_k$

Remarks

- theory is decidable because carrier is finite
- common decision procedures use translation to SAT (bit blasting)
  - eager: no DPLL(T), bit-blast entire formula to SAT problem
  - lazy: second SAT solver as BV theory solver, bit-blast only BV atoms
- solvers heavily rely on preprocessing via rewriting

Definition (Bit Blasting: Formulas)

bit blasting transformation $\mathbf{B}$ transforms BV formula into propositional formula:

$$\mathbf{B}(\varphi \vee \psi) = \mathbf{B}(\varphi) \vee \mathbf{B}(\psi)$$

$$\mathbf{B}(\varphi \wedge \psi) = \mathbf{B}(\varphi) \wedge \mathbf{B}(\psi)$$

$$\mathbf{B}(\neg\varphi) = \neg\mathbf{B}(\varphi)$$

$$\mathbf{B}(t_1 \ \mathit{rel}\ t_2) = \mathbf{B}_r(u_1 \ \mathit{rel}\ u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathbf{B}_t(t_1) = (u_1, \varphi_1) \text{ and } \mathbf{B}_t(t_2) = (u_2, \varphi_2)$$

bit blasting $\mathbf{B}_t$ for term t returns (result u, side condition φ)

$\mathbf{B}_r$ transforms atom into propositional formula

Definition (Bit Blasting: Atoms)
for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ set

- equality

$$\mathbf{B}_r(\mathbf{x}_{k+1} = \mathbf{y}_{k+1}) = (x_k \leftrightarrow y_k) \wedge \cdots \wedge (x_1 \leftrightarrow y_1) \wedge (x_0 \leftrightarrow y_0)$$

- inequality

$$\mathbf{B}_r(\mathbf{x}_{k+1} \neq \mathbf{y}_{k+1}) = (x_k \oplus y_k) \vee \cdots \vee (x_1 \oplus y_1) \vee (x_0 \oplus y_0)$$

- unsigned greater-than or equal

$$\mathbf{B}_r(\mathbf{x}_1 \geqslant_u \mathbf{y}_1) = y_0 \rightarrow x_0$$

$$\mathbf{B}_r(\mathbf{x}_{k+1} \geqslant_u \mathbf{y}_{k+1}) = (x_k \wedge \neg y_k) \vee ((x_k \leftrightarrow y_k) \wedge \mathbf{B}(\mathbf{x}[k-1{:}0] \geqslant_u \mathbf{y}[k-1{:}0]))$$

- unsigned greater-than

$$\mathbf{B}(\mathbf{x}_k >_u \mathbf{y}_k) = \mathbf{B}(\mathbf{x}_k \geqslant_u \mathbf{y}_k) \wedge \mathbf{B}(\mathbf{x}_k \neq \mathbf{y}_k)$$

Definition (Bit Blasting: Bitwise Operations)
for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ use fresh variable $\mathbf{z}_k$ and set

- bitwise and

$$\mathbf{B}_t(\mathbf{x}_k \& \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \wedge y_i)$$

- bitwise or

$$\mathbf{B}_t(\mathbf{x}_k | \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \vee y_i)$$

- bitwise exclusive or

$$\mathbf{B}_t(\mathbf{x}_k \hat{} \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \oplus y_i)$$

- bitwise negation

$$\mathbf{B}_t(-\mathbf{x}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow \neg x_i$$

Definition (Bit Blasting: Addition and Subtraction)

- addition

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = (\mathbf{s}_k, \varphi)$$

where

$$\varphi = (c_0 \leftrightarrow x_0 \wedge y_0) \wedge (s_0 \leftrightarrow x_0 \oplus y_0) \wedge \bigwedge_{i=1}^{k-1} (c_i \leftrightarrow \mathrm{min2}(x_i, y_i, c_{i-1})) \wedge (s_i \leftrightarrow x_i \oplus y_i \oplus c_{i-1})$$

ripple-carry adder:
$\mathbf{c}_k$ are carry bits

for fresh variables $\mathbf{s}_k$ and $\mathbf{c}_k$ and $\mathrm{min2}(a, b, d) = (a \wedge b) \vee (a \wedge d) \vee (b \wedge d)$

- unary minus

$$\mathbf{B}_t(-\mathbf{x}_k) = \mathbf{B}_t(\sim \mathbf{x}_k + \mathbf{1}_k)$$

- subtraction

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = \mathbf{B}_t(\mathbf{x}_k + (-\mathbf{y}_k)$$

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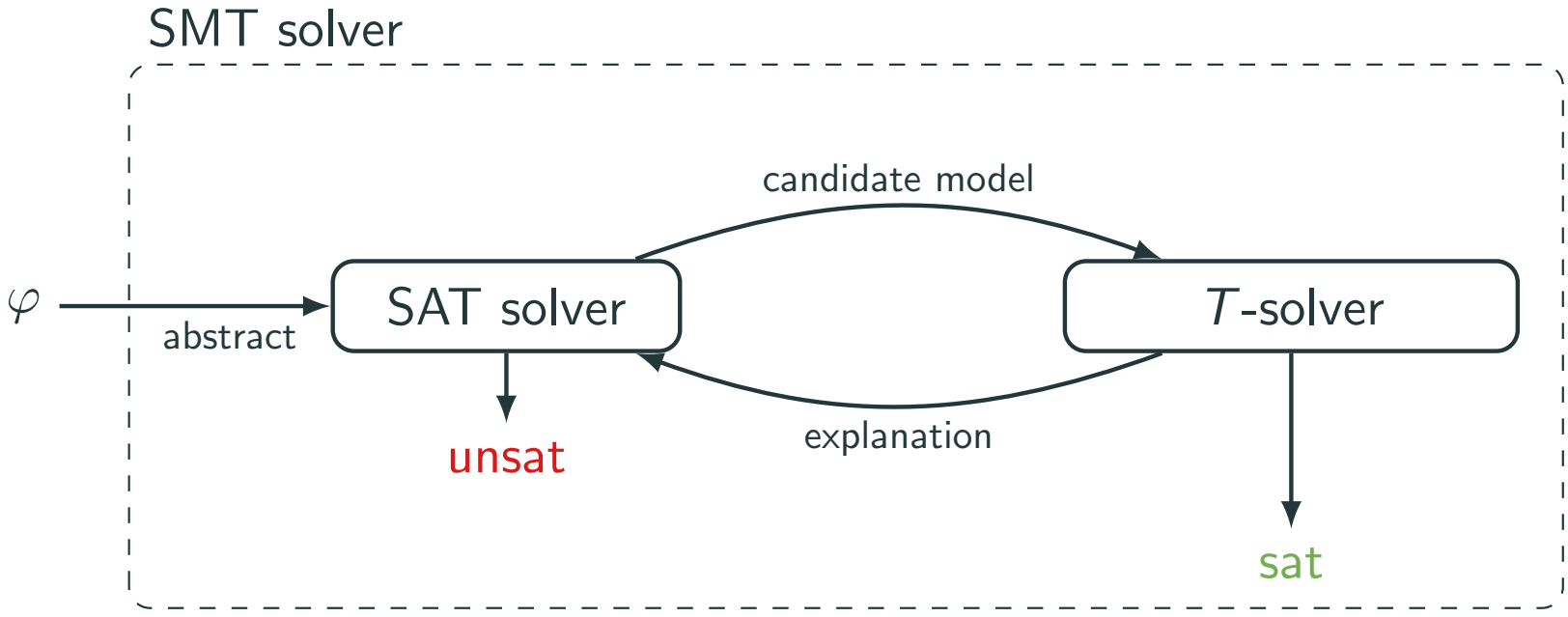

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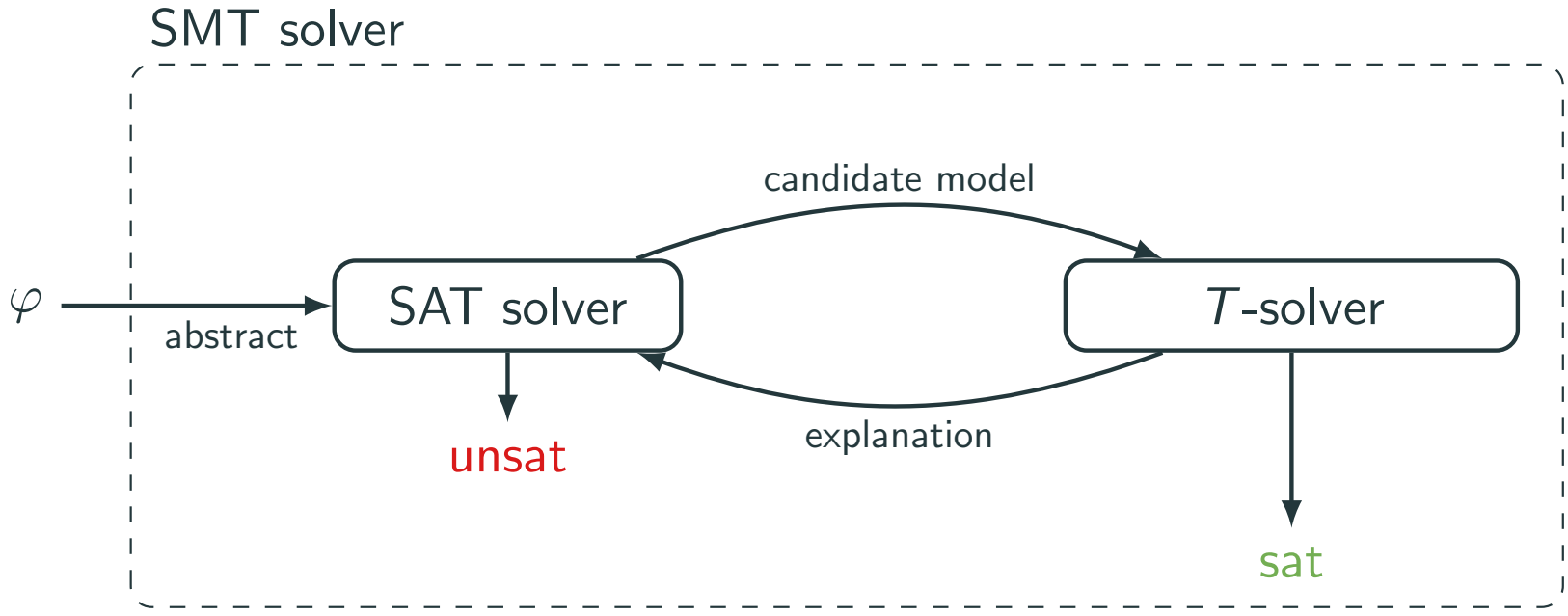

Theory T

- equality logic
- equality + uninterpreted functions (EUF)
- linear arithmetic (LRA and LIA)
- bitvectors (BV)

T-solving method

equality graphs	✓
congruence closure	✓
DPLL(T) Simplex (+ cuts)	✓
bit-blasting	✓

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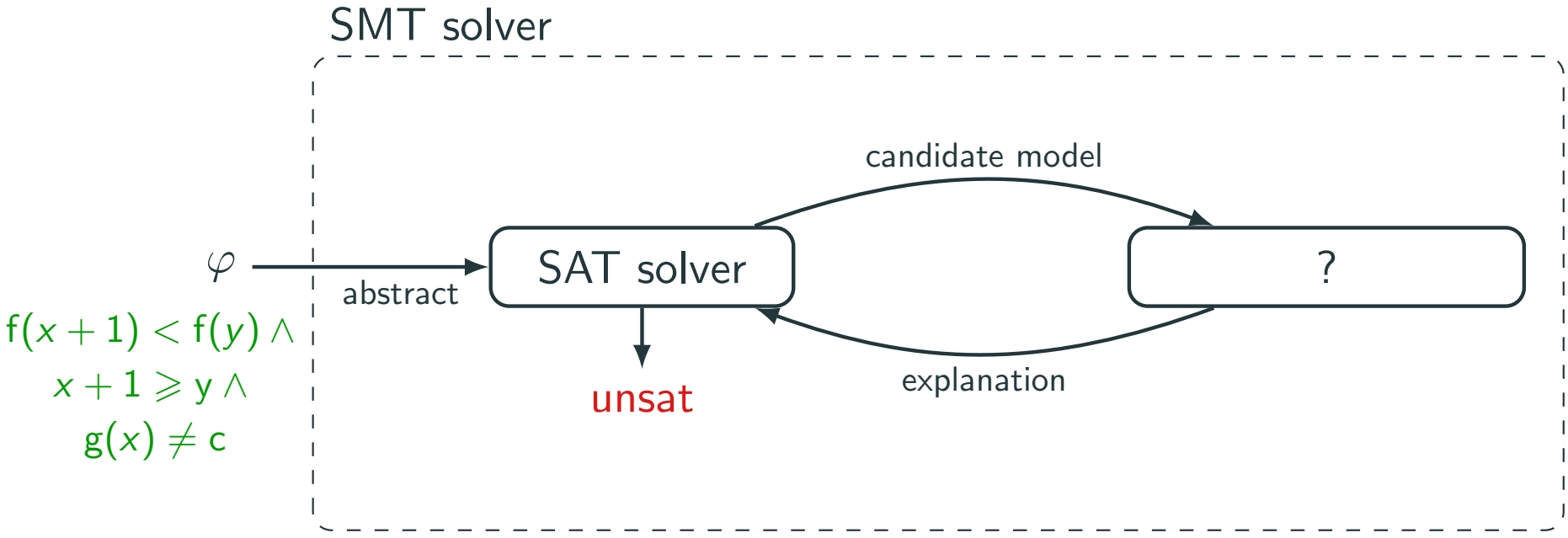

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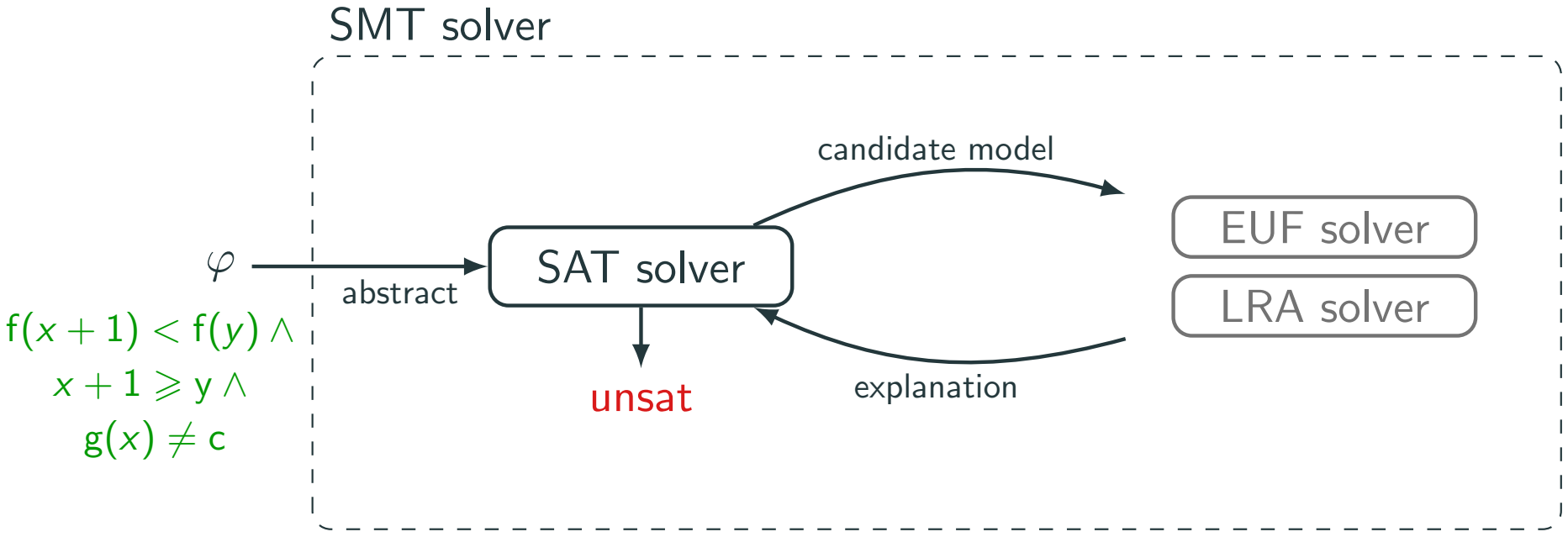

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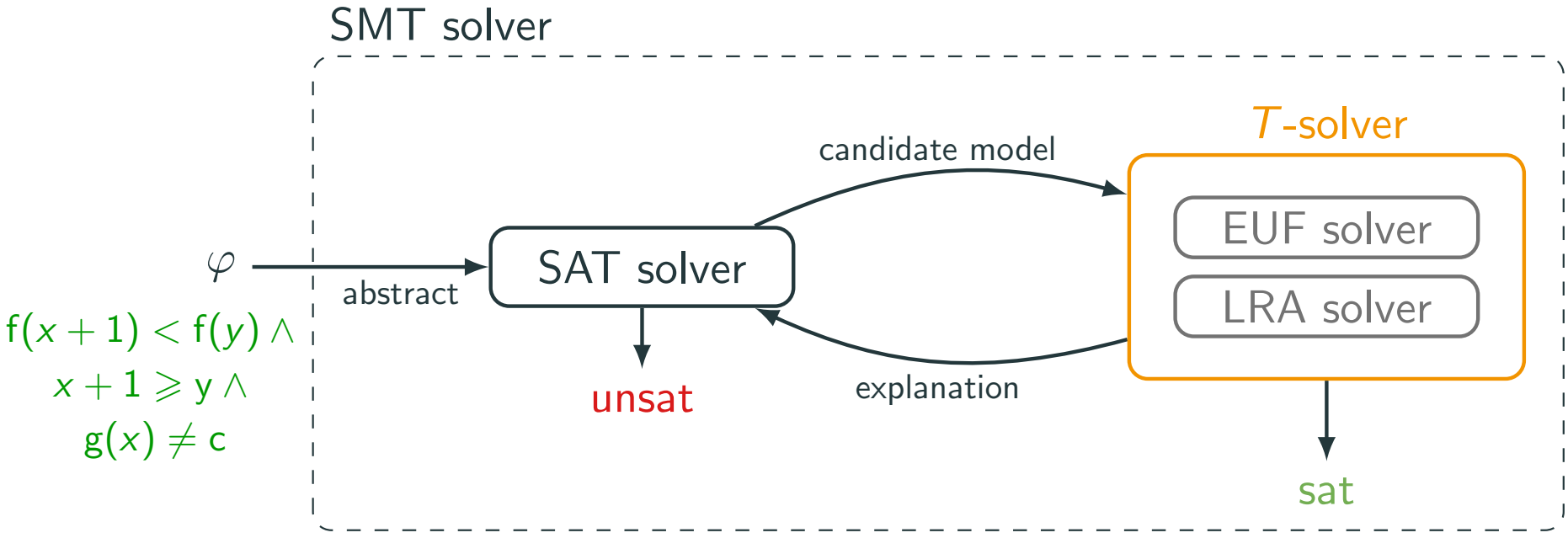

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Theory combinations Nelson-Oppen method

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(first-order) theory T consists of

- signature Σ: set of function and predicate symbols
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Examples

- EUF formula $f(a) = b \wedge f(b) = a$

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- EUF formula $\mathsf{f}(\mathsf{a}) = \mathsf{b} \wedge \mathsf{f}(\mathsf{b}) = \mathsf{a}$
  - has model $\mathcal{M}$ with carrier $\{0, 1\}$, $\mathsf{a}_{\mathcal{M}} = 0$, $\mathsf{b}_{\mathcal{M}} = 1$, $\mathsf{f}_{\mathcal{M}}(x) = \begin{cases} 0 & \text{if } x{=}1 \\ 1 & \text{if } x{=}0 \end{cases}$

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theory combination $T_1 \oplus T_2$ of two theories

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combination of linear arithmetic and uninterpreted functions:

$$x \geqslant y \ \wedge \ y - z \geqslant x \ \wedge \ \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \ \wedge \ z \geqslant 0$$

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$$\varphi \approx \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

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formula φ in combination of LIA and EUF:

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   - guess equivalence relation E on V
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{(x,y)\in E} x = y \quad \wedge \quad \bigwedge_{(x,y)\in V^2\setminus E} x \neq y$$

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- $V = \{x, y, z\}$
- 5 different equivalence relations E, represented by partitionings as:
  1. $\{\{x, y, z\}\}$
  2. $\{\{x, y\}, \{z\}\}$
  3. $\{\{x, z\}, \{y\}\}$
  4. $\{\{x\}, \{y, z\}\}$
  5. $\{\{x\}, \{y\}, \{z\}\}$

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$$\varphi \;\approx\; \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - V is set of shared variables in φ_1 and φ_2
   - guess equivalence relation E on V
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{(x,y)\in E} x = y \;\wedge \bigwedge_{(x,y)\in V^2\setminus E} x \neq y$$

   - if $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

Example

formula φ in combination of LIA and EUF:

$$\underbrace{1 \leqslant x \ \wedge \ x \leqslant 2 \ \wedge \ y = 1 \ \wedge \ z = 2}_{\varphi_1} \wedge \underbrace{\mathsf{f}(x) \neq \mathsf{f}(y) \ \wedge \ \mathsf{f}(x) \neq \mathsf{f}(z)}_{\varphi_2}$$

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$$x + \mathsf{f}(y) = 7 \ \wedge \ x \geqslant 5 \ \wedge \mathsf{f}(y) \geqslant y \ \wedge \ \mathsf{f}(x) \neq \mathsf{f}(y)$$

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Example

formula φ in combination of LIA and EUF:

$$\underbrace{x + z = 7 \wedge x \geqslant 5 \wedge z \geqslant y}_{\varphi_1} \wedge \underbrace{\mathsf{f}(x) \neq \mathsf{f}(y) \wedge \mathsf{f}(y) = z}_{\varphi_2}$$

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- φ is satisfiable, e.g. by $v(x) = 7$, $v(y) = v(z) = 0$, and $\mathsf{f}_{\mathcal{M}}(x) = x$

Fact

number of equivalence relations is given by Bell numbers: very inefficient

Outline

- Summary of Last Week
- Nelson-Oppen Combination Method
  - Nondeterministic Version
  - Deterministic Version
- Application: Collision Attacks

Definition

theory T is convex if

$$F \models_T \bigvee_{i=1}^{n} u_i = v_i \qquad \text{implies} \qquad F \models_T u_i = v_i \quad \text{for some } 1 \leqslant i \leqslant n$$

for every conjunction of literals F and variables $u_1, \ldots, u_n, v_1, \ldots, v_n$

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- EUF is convex:

$$\mathsf{f}(\mathsf{a}) = x \wedge \mathsf{f}(\mathsf{b}) = y \wedge \mathsf{f}(\mathsf{c}) = z \wedge \mathsf{a} = \mathsf{b} \wedge \mathsf{b} = \mathsf{c} \quad \models_T \quad x = y \vee x = z$$

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Nelson-Oppen Method: Deterministic Version

Input	quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories T_1 and T_2
Output	satisfiable or unsatisfiable

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Example (Nelson-Oppen, deterministic)

consider φ over combination of LRA and EUF:

$$x \geqslant y \ \wedge \ y - z \geqslant x \ \wedge \ \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \ \wedge \ z \geqslant 0$$

- purify φ

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satisfiable $\qquad\qquad \varphi_1 \ \wedge \ E \ \longrightarrow \ x = y$

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consider φ over combination of LRA and EUF:

$$x \geqslant y \ \wedge \ y - z \geqslant x \ \wedge \ \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \ \wedge \ z \geqslant 0$$

- purify φ

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Example

consider φ over combination of EUF and BV (not stably infinite):

$$\bigwedge_{1 \leqslant i \leqslant 5} \bigwedge_{i < j \leqslant 5} \mathsf{f}(x_i) \neq \mathsf{f}(x_j)$$

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approaches exist to combine non-stably infinite theories:

- using concept of shiny theories (link)
- using concept of polite theories (link)

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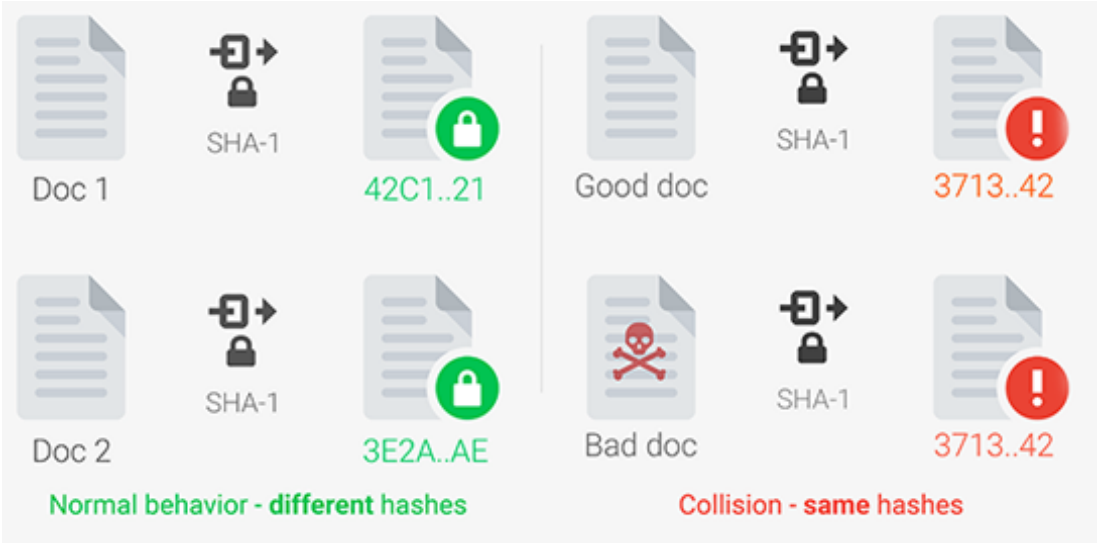

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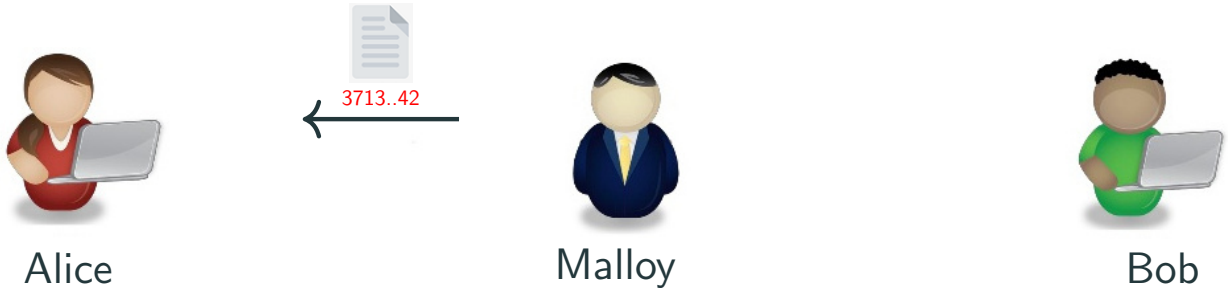

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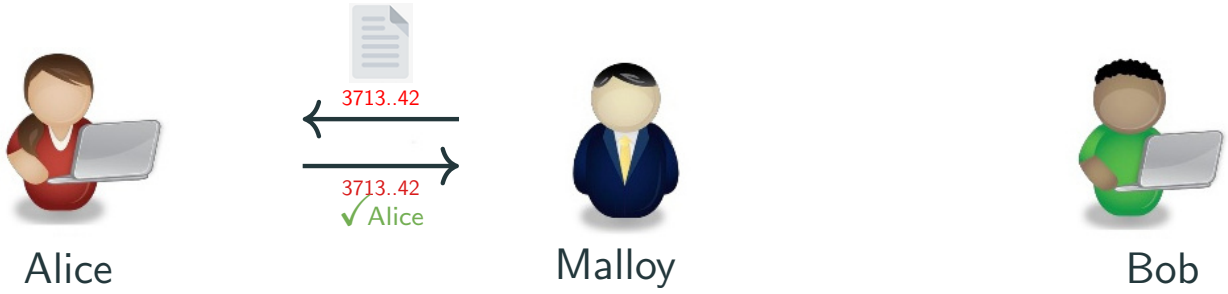

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Extension: Chosen-Prefix Collision Attack

find values x and y such that $\forall m_1\ m_2.\ f(x \cdot m_1) = f(y \cdot m_2)$

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SHA-0, SHA-1, SHA-256, MD5, MD6, BLAKE2, RIPEMD-160, ...

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Collision Attack: Shift-Add-Xor Hash

- widely used non-cryptographic string hash function
- given string s, compute hash $\texttt{sax}(s)$

```
unsigned sax(char *s, int len){
    unsigned h = 0;
    for (int i = 0; i < len; i++)
        h = h ^ ((h << 5) + (h >> 2) + s[i]);
    return h;
}
```

- collision attack: `sax_collision.py`

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Tools for SAT/SMT-Based Cryptanalysis

- CryptoMiniSat
- CryptoSMT
- Transalg
- ...

Bibliography

Greg Nelson and Derek C. Oppen

Simplification by Cooperating Decision Procedures

ACM Transactions on Programming Languages and Systems 2(1), pp 245–257, 1979.

Nuno P. Lopes and José Monteiro.

Automatic equivalence checking of programs with uninterpreted functions and integer arithmetic.

International Journal on Software Tools for Technology Transfer 18(4), pp 359–374, 2016.