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SAT and SMT Solving

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Outline

- Summary of Last Week
- Nelson-Oppen Combination Method
- Application: Collision Attacks

Definition (Bit Vector Theory)

- variable $\mathbf{x}_k$ is list of length k of propositional variables $x_{k-1} \ldots x_2 x_1 x_0$
- constant n_k is bit list of length k
- formulas built according to grammar

$$
\begin{aligned}
\mathit{formula} &:= (\mathit{formula} \vee \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\neg \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &:= \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{true} \mid \mathit{false} \\
\mathit{rel} &:= \ = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &:= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{var} \mid \mathit{constant} \mid \mathit{term}[i{:}j] \mid \\
&\qquad (\mathit{formula}\ ?\ \mathit{term} : \mathit{term}) \\
\mathit{binop} &:= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid | \mid \hat{}\ \mid :: \\
\mathit{unop} &:= \sim \mid -
\end{aligned}
$$

- axioms are equality axioms plus rules for arithmetic/comparison/bitwise operations on bit vectors of length k
- solution assigns bit list of length k to variables $\mathbf{x}_k$

Remarks

- theory is decidable because carrier is finite
- common decision procedures use translation to SAT (bit blasting)
  - eager: no DPLL(T), bit-blast entire formula to SAT problem
  - lazy: second SAT solver as BV theory solver, bit-blast only BV atoms
- solvers heavily rely on preprocessing via rewriting

Definition (Bit Blasting: Formulas)

bit blasting transformation $\mathbf{B}$ transforms BV formula into propositional formula:

$$\mathbf{B}(\varphi \vee \psi) = \mathbf{B}(\varphi) \vee \mathbf{B}(\psi)$$

$$\mathbf{B}(\varphi \wedge \psi) = \mathbf{B}(\varphi) \wedge \mathbf{B}(\psi)$$

$$\mathbf{B}(\neg\varphi) = \neg\mathbf{B}(\varphi)$$

$$\mathbf{B}(t_1 \ \mathit{rel} \ t_2) = \mathbf{B}_r(u_1 \ \mathit{rel} \ u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathbf{B}_t(t_1) = (u_1, \varphi_1) \text{ and } \mathbf{B}_t(t_2) = (u_2, \varphi_2)$$

bit blasting $\mathbf{B}_t$ for term t returns (result u, side condition φ)

$\mathbf{B}_r$ transforms atom into propositional formula

Definition (Bit Blasting: Atoms)
for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ set

- equality

$$\mathbf{B}_r(\mathbf{x}_{k+1} = \mathbf{y}_{k+1}) = (x_k \leftrightarrow y_k) \wedge \cdots \wedge (x_1 \leftrightarrow y_1) \wedge (x_0 \leftrightarrow y_0)$$

- inequality

$$\mathbf{B}_r(\mathbf{x}_{k+1} \neq \mathbf{y}_{k+1}) = (x_k \oplus y_k) \vee \cdots \vee (x_1 \oplus y_1) \vee (x_0 \oplus y_0)$$

- unsigned greater-than or equal

$$\mathbf{B}_r(\mathbf{x}_1 \geqslant_u \mathbf{y}_1) = y_0 \to x_0$$

$$\mathbf{B}_r(\mathbf{x}_{k+1} \geqslant_u \mathbf{y}_{k+1}) = (x_k \wedge \neg y_k) \vee ((x_k \leftrightarrow y_k) \wedge \mathbf{B}(\mathbf{x}[k-1{:}0] \geqslant_u \mathbf{y}[k-1{:}0]))$$

- unsigned greater-than

$$\mathbf{B}(\mathbf{x}_k >_u \mathbf{y}_k) = \mathbf{B}(\mathbf{x}_k \geqslant_u \mathbf{y}_k) \wedge \mathbf{B}(\mathbf{x}_k \neq \mathbf{y}_k)$$

Definition (Bit Blasting: Bitwise Operations)

for bit vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ use fresh variable $\mathbf{z}_k$ and set

- bitwise and

$$\mathbf{B}_t(\mathbf{x}_k \& \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \wedge y_i)$$

- bitwise or

$$\mathbf{B}_t(\mathbf{x}_k | \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \vee y_i)$$

- bitwise exclusive or

$$\mathbf{B}_t(\mathbf{x}_k \;\hat{}\; \mathbf{y}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow (x_i \oplus y_i)$$

- bitwise negation

$$\mathbf{B}_t(-\mathbf{x}_k) = (\mathbf{z}_k, \varphi) \qquad \varphi = \bigwedge_{i=0}^{k-1} z_i \leftrightarrow \neg x_i$$

Definition (Bit Blasting: Addition and Subtraction)

- addition

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = (\mathbf{s}_k, \varphi)$$

ripple-carry adder:
$\mathbf{c}_k$ are carry bits

where

$$\varphi = (c_0 \leftrightarrow x_0 \wedge y_0) \wedge (s_0 \leftrightarrow x_0 \oplus y_0) \wedge$$
$$\bigwedge_{i=1}^{k-1} (c_i \leftrightarrow \min2(x_i, y_i, c_{i-1})) \wedge (s_i \leftrightarrow x_i \oplus y_i \oplus c_{i-1})$$

for fresh variables $\mathbf{s}_k$ and $\mathbf{c}_k$ and $\min2(a, b, d) = (a \wedge b) \vee (a \wedge d) \vee (b \wedge d)$

- unary minus

$$\mathbf{B}_t(-\mathbf{x}_k) = \mathbf{B}_t(\sim \mathbf{x}_k + \mathbf{1}_k)$$

- subtraction

$$\mathbf{B}_t(\mathbf{x}_k + \mathbf{y}_k) = \mathbf{B}_t(\mathbf{x}_k + (-\mathbf{y}_k))$$

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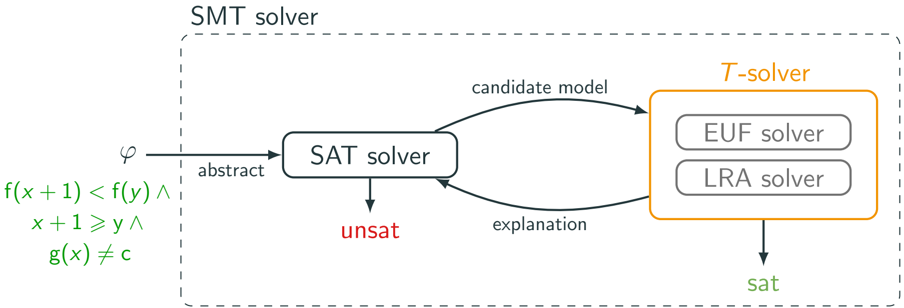
Lazy SMT Solving



Theory T	**T-solving method**	
▶ equality logic	equality graphs	✓
▶ equality + uninterpreted functions (EUF)	congruence closure	✓
▶ linear arithmetic (LRA and LIA)	DPLL(T) Simplex (+ cuts)	✓
▶ bitvectors (BV)	bit-blasting	✓

Theory combinations Nelson-Oppen method

Definition

(first-order) theory T consists of

- signature Σ: set of function and predicate symbols
- axioms $\mathcal{A}$: set of sentences in first-order logic over Σ

Definition

theory is stably infinite if every satisfiable quantifier-free formula has model with infinite carrier set

Facts

- linear arithmetic (LIA, LRA) is stably infinite
- equality + uninterpreted functions (EUF) is stably infinite
- bit vector theory (BV) is not stably infinite

all models are infinite

all models are finite

Examples

- EUF formula $\mathsf{f}(\mathsf{a}) = \mathsf{b} \wedge \mathsf{f}(\mathsf{b}) = \mathsf{a}$
  - has model $\mathcal{M}$ with carrier $\{0,1\}$, $\mathsf{a}_{\mathcal{M}} = 0$, $\mathsf{b}_{\mathcal{M}} = 1$, $\mathsf{f}_{\mathcal{M}}(x) = \begin{cases} 0 & \text{if } x{=}1 \\ 1 & \text{if } x{=}0 \end{cases}$
  - has model $\mathcal{M}'$ with carrier $\mathbb{Z}$, $\mathsf{a}_{\mathcal{M}'} = -1$, $\mathsf{b}_{\mathcal{M}'} = 1$ and $\mathsf{f}_{\mathcal{M}'}(x) = -x$
- theory with $\Sigma = \{a, b, =\}$ and $\mathcal{A} = \{\forall x\ (x = a \vee x = b)\} \cup \mathcal{A}_{=}$ is not stably infinite: has only finite models!

Definition

theory combination $T_1 \oplus T_2$ of two theories

- T_1 over signature Σ_1 with axioms $\mathcal{A}_1$
- T_2 over signature Σ_2 with axioms $\mathcal{A}_2$

has signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$

Example

combination of linear arithmetic and uninterpreted functions:

$$x \geqslant y \;\wedge\; y - z \geqslant x \;\wedge\; \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \;\wedge\; z \geqslant 0$$

Assumptions

two stably infinite theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

such that

- $\Sigma_1 \cap \Sigma_2 = \{=\}$
- T_1-satisfiability of quantifier-free Σ_1-formulas is decidable
- T_2-satisfiability of quantifier-free Σ_2-formulas is decidable

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Nelson-Oppen Method: Nondeterministic Version

input: quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$

output: satisfiable or unsatisfiable

1. purification

$$\varphi \;\approx\; \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - V is set of shared variables in φ_1 and φ_2
   - guess equivalence relation E on V
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{(x,y)\in E} x = y \quad \wedge \quad \bigwedge_{(x,y)\in V^2 \setminus E} x \neq y$$

   - if $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

Example

formula φ in combination of LIA and EUF:

$$\underbrace{1 \leqslant x \ \wedge \ x \leqslant 2 \ \wedge \ y = 1 \ \wedge \ z = 2}_{\varphi_1} \wedge \underbrace{\mathsf{f}(x) \neq \mathsf{f}(y) \ \wedge \ \mathsf{f}(x) \neq \mathsf{f}(z)}_{\varphi_2}$$

- $V = \{x, y, z\}$
- 5 different equivalence relations E, represented by partitionings as:
  1. $\{\{x, y, z\}\}$ $\quad \alpha(V, E) = x{=}y \wedge y{=}z \wedge x{=}z$
     $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
  2. $\{\{x, y\}, \{z\}\}$ $\quad \alpha(V, E) = x{=}y \wedge y{\neq}z \wedge x{\neq}z$
     $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  3. $\{\{x, z\}, \{y\}\}$ $\quad \alpha(V, E) = x{=}z \wedge x{\neq}y \wedge z{\neq}y$
     $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  4. $\{\{x\}, \{y, z\}\}$ $\quad \alpha(V, E) = y{=}z \wedge x{\neq}y \wedge x{\neq}z$
     $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
  5. $\{\{x\}, \{y\}, \{z\}\}$ $\quad \alpha(V, E) = x{\neq}y \wedge y{\neq}z \wedge x{\neq}z$
     $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
- φ is unsatisfiable

Example

formula φ in combination of LIA and EUF:

$$\underbrace{x + z = 7 \ \wedge\ x \geqslant 5 \ \wedge\ z \geqslant y}_{\varphi_1} \ \wedge\ \underbrace{\mathsf{f}(x) \neq \mathsf{f}(y) \ \wedge\ \mathsf{f}(y) = z}_{\varphi_2}$$

- $V = \{x, y, z\}$
- 5 different equivalence relations E, represented by partitionings as:
  1. $\{\{x, y, z\}\}$ $\quad \alpha(V, E) = x{=}y \wedge y{=}z \wedge x{=}z$
     $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
  2. $\{\{x, y\}, \{z\}\}$ $\quad \alpha(V, E) = x{=}y \wedge y{\neq}z \wedge x{\neq}z$
     $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  3. $\{\{x, z\}, \{y\}\}$ $\quad \alpha(V, E) = x{=}z \wedge x{\neq}y \wedge z{\neq}y$
     $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  4. $\{\{x\}, \{y, z\}\}$ $\quad \alpha(V, E) = y{=}z \wedge x{\neq}y \wedge x{\neq}z$
     $\varphi_1 \wedge \alpha(V, E)$ is satisfiable
  5. $\{\{x\}, \{y\}, \{z\}\}$ $\quad \alpha(V, E) = x{\neq}y \wedge y{\neq}z \wedge x{\neq}z$
- φ is satisfiable, e.g. by $v(x) = 7$, $v(y) = v(z) = 0$, and $\mathsf{f}_{\mathcal{M}}(x) = x$

Fact

number of equivalence relations is given by Bell numbers: very inefficient

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Definition

theory T is convex if

$$F \models_T \bigvee_{i=1}^{n} u_i = v_i \qquad \text{implies} \qquad F \models_T u_i = v_i \quad \text{for some } 1 \leqslant i \leqslant n$$

for every conjunction of literals F and variables $u_1, \dots, u_n, v_1, \dots, v_n$

Facts

- linear arithmetic over integers (LIA) is not convex
- linear arithmetic over rationals (LRA) is convex
- equality logic with uninterpreted functions (EUF) is convex

Example

- LIA is not convex:

$$1 \leqslant x \leqslant 2 \ \wedge\ y = 1 \ \wedge\ z = 2 \quad \models_T \quad x = y \ \vee\ x = z$$

but

$$1 \leqslant x \leqslant 2 \ \wedge\ y = 1 \ \wedge\ z = 2 \quad \not\models_T \quad x = y$$

$$1 \leqslant x \leqslant 2 \ \wedge\ y = 1 \ \wedge\ z = 2 \quad \not\models_T \quad x = z$$

- EUF is convex:

$$\mathrm{f(a)} = x \wedge \mathrm{f(b)} = y \wedge \mathrm{f(c)} = z \wedge \mathrm{a} = \mathrm{b} \wedge \mathrm{b} = \mathrm{c} \quad \models_T \quad x = y \vee x = z$$

and

$$\mathrm{f(a)} = x \wedge \mathrm{f(b)} = y \wedge \mathrm{f(c)} = z \wedge \mathrm{a} = \mathrm{b} \wedge \mathrm{b} = \mathrm{c} \quad \models_T \quad x = y$$

Nelson-Oppen Method: Deterministic Version

Input quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories T_1 and T_2

Output satisfiable or unsatisfiable

1. purification $\varphi \approx \varphi_1 \wedge \varphi_2$ for Σ_1-formula φ_1 and Σ_2-formula φ_2
2. V: set of shared variables in φ_1 and φ_2

   E: discovered equalities between variables in V (initially $E = \varnothing$)
3. test satisfiability of $\varphi_1 \wedge E$
   - if $\varphi_1 \wedge E$ is T_1-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
4. test satisfiability of $\varphi_2 \wedge E$
   - if $\varphi_2 \wedge E$ is T_2-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
5. if E has been extended in steps 3 or 4 then go to step 3

   else return satisfiable

Example (Nelson-Oppen, deterministic)

consider φ over combination of LRA and EUF:

$$x \geqslant y \;\wedge\; y - z \geqslant x \;\wedge\; \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \;\wedge\; z \geqslant 0$$

- purify φ:

φ_1: $x \geqslant y \;\wedge\; y - z \geqslant x$

φ_2: $\mathsf{f}(u) \neq \mathsf{f}(z) \;\wedge\; v = \mathsf{f}$

test all (finitely many) equations,
or use T-propagation

- implied equalities between shared variables:

$$E\colon\; x = y \;\wedge\; v = w \;\wedge\; z = u$$

- test satisfiability of $\varphi_2 \wedge E$ in EUF and compute implied equalities

satisfiable $\qquad \varphi_2 \;\wedge\; E \;\longrightarrow\; z = u$

- φ is unsatisfiable

Remark

deterministic Nelson-Oppen procedure can be extended to non-convex theories:
do case-splitting for implied disjunction of equalities

Example

consider φ over combination of LIA and EUF:

$$1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(1) \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(2)$$

- purify φ:

$$\varphi_1: \ 1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ w_1 = 1 \ \wedge\ w_2 = 2$$

$$\varphi_2: \ \mathsf{f}(x) \neq \mathsf{f}(w_1) \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(w_2)$$

- implied equalities:

$$E: \ x = w_2$$

- test satisfiability of $\varphi_2 \wedge E$ in EUF, compute (disjunction of) equalities:

unsatisfiable $\quad \varphi_2 \ \wedge\ E \ \longrightarrow\ \bot$

- case split: $x = w_1$ or $x = w_2$
- φ is unsatisfiable

Example

consider φ over combination of EUF and BV (not stably infinite):

$$\bigwedge_{1 \leqslant i \leqslant 5} \bigwedge_{i < j \leqslant 5} \mathsf{f}(x_i) \neq \mathsf{f}(x_j)$$

for variables $x_1, \ldots, x_5$ of bitvector type with two bits

- φ is already pure:
  - EUF formula $\varphi_1 = \varphi$
  - BV formula $\varphi_2 = \top$
- there are no shared variables
- Nelson-Oppen concludes satisfiability
  - deterministic version: no implied equalities
  - non-deterministic version: usually equivalence relations consider only shared variables*

Remark

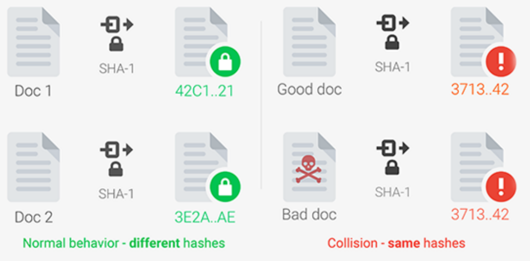
approaches exist to combine non-stably infinite theories:

- using concept of shiny theories (link)
- using concept of polite theories (link)

*In this example, unsatisfiability could be detected if all equivalence relations among all variables are checked, but even this does not help if the counterexample is done for theory of arrays.

Cryptographic Hash Functions

- one-way function: maps arbitrary data to bit string of fixed size (hash)
- considered infeasible to invert, and to find messages with same hash
- problem: hash collisions



Classical Collision Attack Scenario

- Malloy wants to send malicious document to Bob pretending it be from Alice (currently) practically infeasible to invert

Example (Cryptographic hash functions)

SHA-0, SHA-1, SHA-256, MD5, MD6, BLAKE2, RIPEMD-160, ...

Collision Attack: Shift-Add-Xor Hash

- widely used non-cryptographic string hash function
- given string s, compute hash `sax`(s)

```
unsigned sax(char *s, int len){
    unsigned h = 0;
    for (int i = 0; i < len; i++)
        h = h ^ ((h << 5) + (h >> 2) + s[i]);
    return h;
}
```

- collision attack: `sax_collision.py`

More Cryptanalysis using SAT/SMT

- collision attacks (preimage attacks) for current hash functions such as MD4, MD5, SHA-256, CryptoHash, Keccak, ...
- exhibit classes of weak keys (or prove their absence)
  for block ciphers such as IDEA, WIDEA-*n*, or MESH-8
- solve inversion problems, e.g. for 20 bit DES key
- reason about crypto primitives
- help prove complexity bounds of certain operations

Tools for SAT/SMT-Based Cryptanalysis

- CryptoMiniSat
- CryptoSMT
- Transalg
- ...

Bibliography

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