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# SAT and SMT Solving


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lecture 12
WS 2022

# Outline

- Summary of Last Week
- Bounded Model Checking for Verification
- Quantifiers for SMT

## Definitions

- theory consists of
  - signature $\Sigma$: set of function and predicate symbols
  - axioms $T$: set of sentences in first-order logic in which only function and predicate symbols of $\Sigma$ appear
- theory is stably infinite if every satisfiable quantifier-free formula has model with infinite carrier set
- theory $T$ is convex if $F \vDash_T \bigvee_{i=1}^{n} u_i = v_i$ implies $F \vDash_T u_i = v_i$ for some $1 \leqslant i \leqslant n$ $\forall$ quantifier-free conjunction $F$ and variables $u_i, v_i$

## Definition

theory combination $T_1 \oplus T_2$: signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$

## Assumptions

two stably infinite theories $T_1$, $T_2$ over signatures $\Sigma_1$, $\Sigma_2$ such that

- $\Sigma_1 \cap \Sigma_2 = \{=\}$
- $T_i$-satisfiability of quantifier-free $\Sigma_i$-formulas is decidable for $i \in \{1, 2\}$

## Nelson-Oppen Method: Nondeterministic Version

Input quantifier-free conjunction $\varphi$ in theory combination $T_1 \oplus T_2$
*Output* satisfiable or unsatisfiable

1. purification

$$\varphi \approx \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - $V$ is set of shared variables in $\varphi_1$ and $\varphi_2$
   - guess equivalence relation $E$ on $V$
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{x\,E\,y} x = y \quad \wedge \bigwedge_{\neg(x\,E\,y)} x \neq y$$

   - if $\varphi_1 \wedge \alpha(V, E)$ is $T_1$-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is $T_2$-satisfiable then return satisfiable else return unsatisfiable

## Nelson-Oppen Method: Deterministic Version

Input quantifier-free conjunction $\varphi$ in combination $T_1 \oplus T_2$ of convex theories $T_1$ and $T_2$

Output satisfiable or unsatisfiable

1. purification $\varphi \approx \varphi_1 \wedge \varphi_2$ for $\Sigma_1$-formula $\varphi_1$ and $\Sigma_2$-formula $\varphi_2$
2. $V$: set of shared variables in $\varphi_1$ and $\varphi_2$

   $E$: already discovered equalities between variables in $V$
3. test satisfiability of $\varphi_1 \wedge E$ (and add implied equations)
   - if $\varphi_1 \wedge E$ is $T_1$-unsatisfiable then return unsatisfiable
   - else add new implied equalities to $E$
4. test satisfiability of $\varphi_2 \wedge E$ (and add implied equations)
   - if $\varphi_2 \wedge E$ is $T_2$-unsatisfiable then return unsatisfiable
   - else add new implied equalities to $E$
5. if $E$ has been extended in steps 3 or 4 then go to step 2

   else return satisfiable

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# Disastrous Software Bugs

## Ariane 5 Flight 501 (1996)

- destroyed 37 seconds after launch
- software for Ariane 4 for was reused
- software error: data conversion from 64-bit floating point to 16-bit integer caused arithmetic overflow
- cost: 370 million $

http://en.wikipedia.org/wiki/Ariane_5_Flight_501

## Mars Exploration Rover "Spirit" (2004)

- landed on January 4
- stopped communicating on January 21
- software error: stuck in reboot loop
- reboot failed because of flash memory failure, ultimate problem: too many files

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## Heathrow Terminal 5 Opening (2008)


- baggage system collapsed on opening day
- 42,000 bags not shipped with their owners, 500 flights cancelled
- software was tested but did not work properly with real-world load
- cost 50 million £

http://www.zdnet.com/article/it-failure-at-heathrow-t5-what-really-happened

## Trading Glitch at Knight Capital (2012)

- bug in trading software resulted in 45 minutes of uncontrolled buys
- company did 11% of US trading that year
- software was run in invalid configuration
- 440 million $ lost

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## Death in Self-Driving Car Crash (2018)

- person died in accident with Uber's self-driving car
- victim was wrongly classified by software as non-obstacle

http://www.siliconrepublic.com/companies/uber-bug-crash

## Software is Ubiquitous in Critical Systems

transport, energy, medicine, communication, finance, embedded systems, ...

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## How to Ensure Correctness of Software?

- testing
  - \+ cheap, simple
  - – checks desired result only for given set of testcases
- verification
  - \+ can prove automatically that system meets specification, i.e., desired output is delivered for all inputs
  - – more costly

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- widely used verification approach to
  - find bugs in software and hardware
  - prove correctness of models

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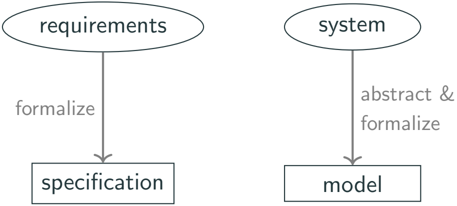
- bounded model checking can be reduced to SAT/SMT

# Model Checking: Workflow



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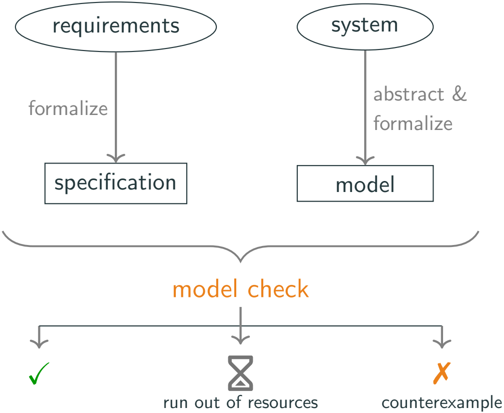

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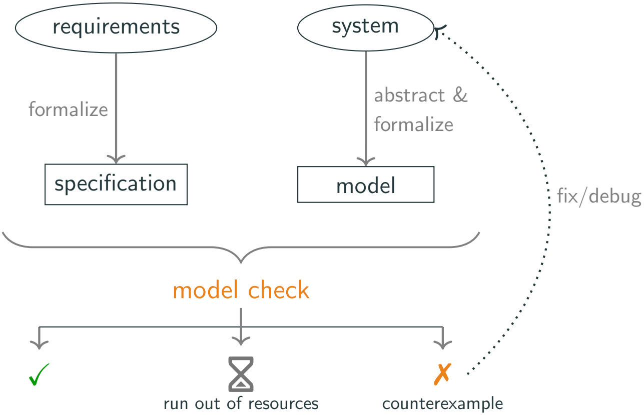

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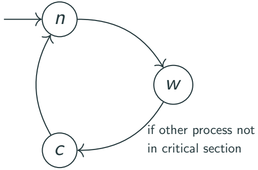
- process can be abstracted to model $\mathcal{M} = \langle S, R \rangle$ with states $S = \{n, w, c\}$ and transitions $R$:



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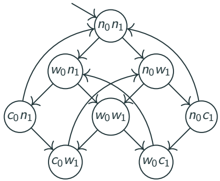

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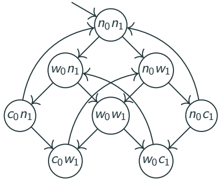


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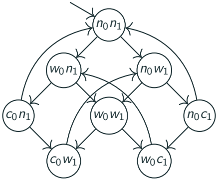


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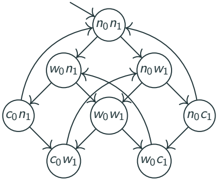


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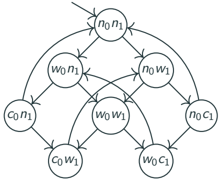


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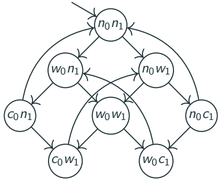


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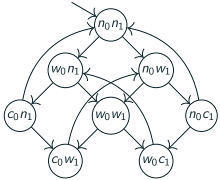


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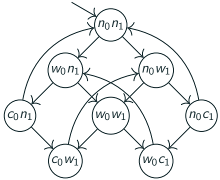


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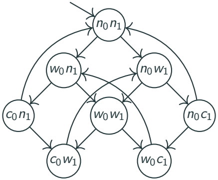


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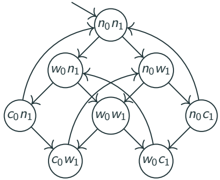


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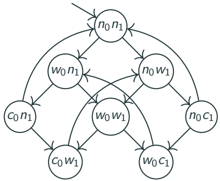


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### Observation

model checking is feasible for this example because state space is finite and small

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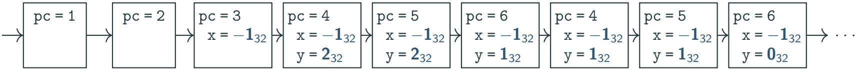
- (part of) model:



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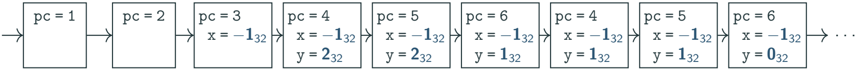

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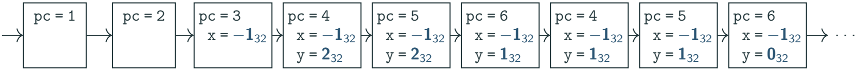

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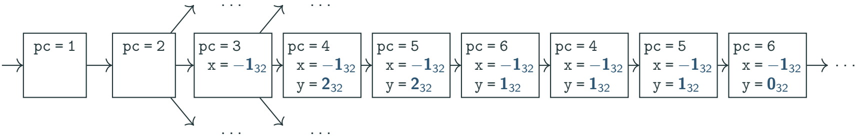


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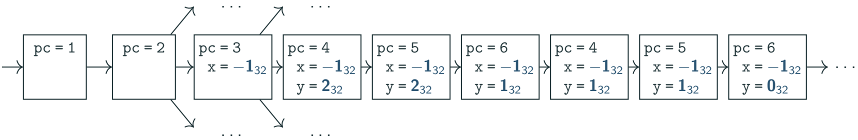


- but state space is very large: $(2^{32})^2 \cdot 7$ for bit width 32
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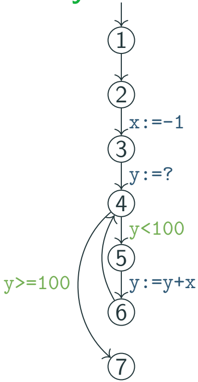

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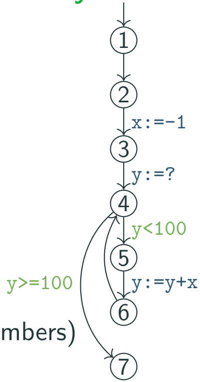


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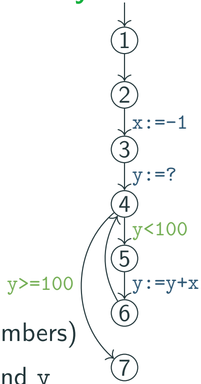


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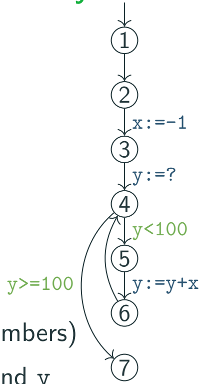


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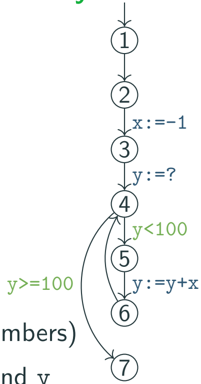


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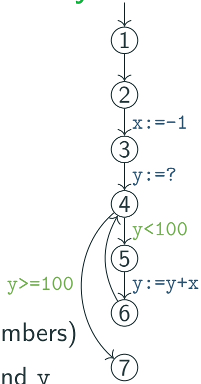


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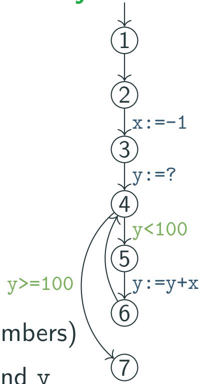


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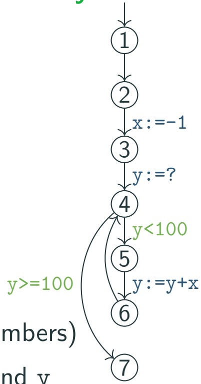


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# Idea

consider symbolic program executions with bounded length,
try to solve with SMT solver

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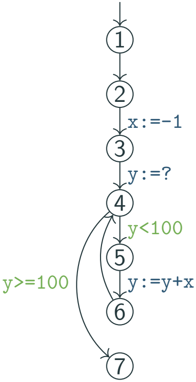
   - $I(\langle pc, x, y \rangle) = (pc = 1)$ to characterize initial state



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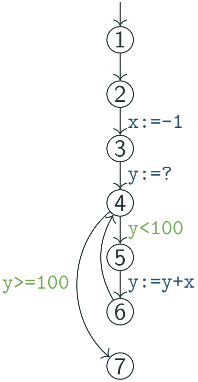

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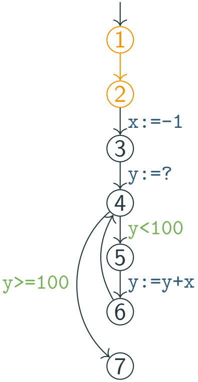

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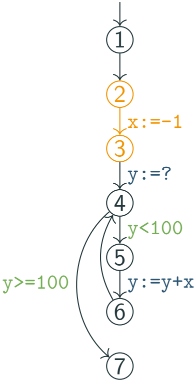

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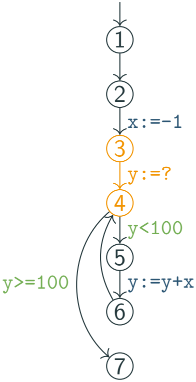

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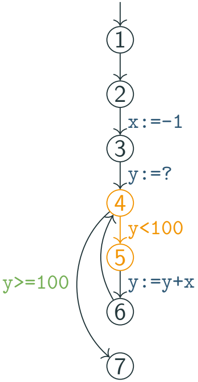

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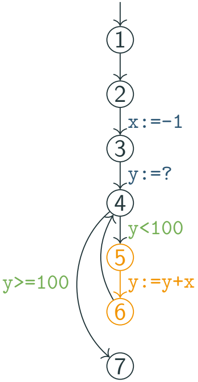

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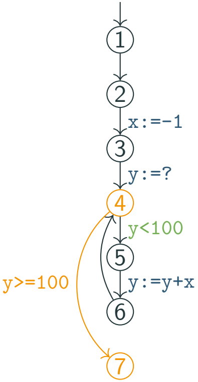

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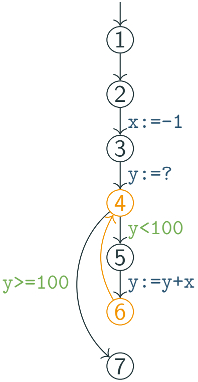

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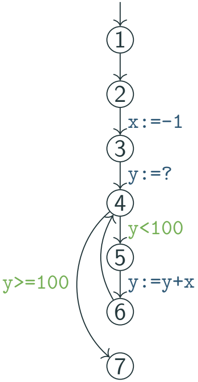

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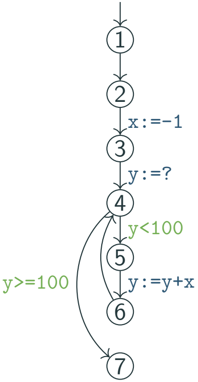


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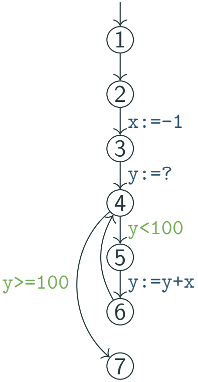


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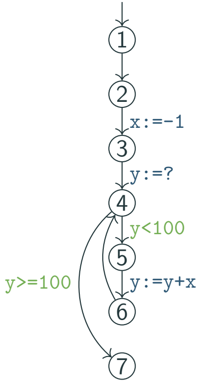


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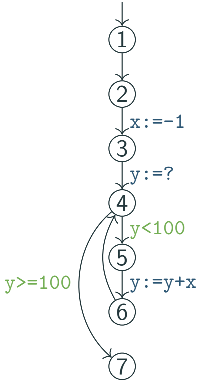


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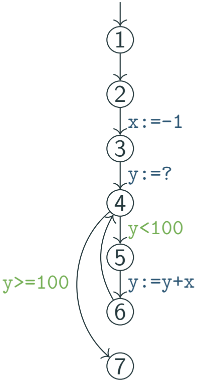


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transition system $\mathcal{T} = (S, \rightarrow, S_0, L)$ where

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## Remark

$S$ and $A$ may be (countably) infinite

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given transition system and property $\mathsf{G}\ \psi$, look for counterexamples in $\leqslant k$ steps

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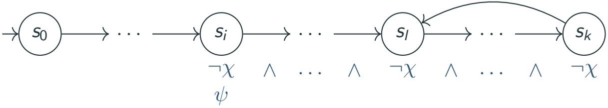

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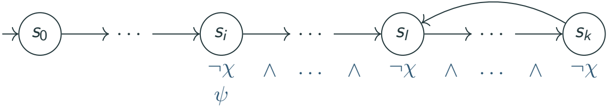


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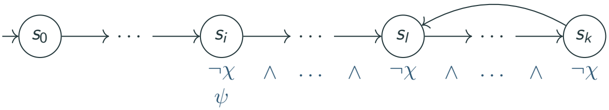


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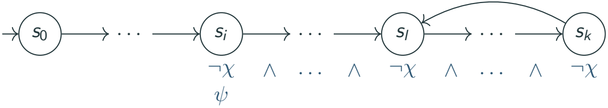


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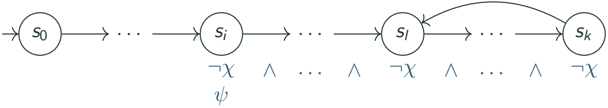


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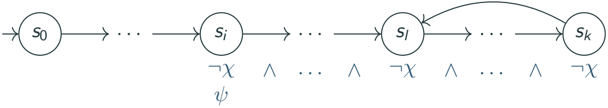


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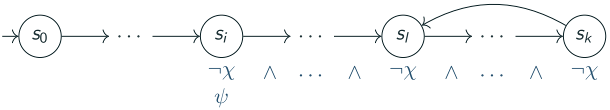


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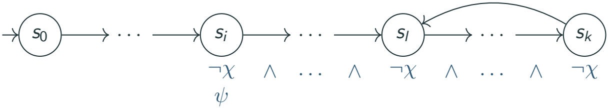


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- there is step $s \to s'$ for $s = \langle pc, v_0, \ldots, v_n \rangle$ and $s' = \langle pc', v_0', \ldots, v_n' \rangle$ iff $P$ admits step from $s$ to $s'$
- $S_0$ consists of initial program states
- atom set $A$ consists of all propositional formulas over $pc, v_0, \ldots, v_n$
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## Program Graph

- nodes are line numbers
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## Transition System $\mathcal{T}(P)$ from Program $P$

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- program graph is useful to derive encoding of $\mathcal{T}(P)$

## Checking an Explicit Assertion

```c
int n;
int main() {
  int i=0, j=10, s=0;
  for(i=0; i<=n; i++) {
    if (i<j)
      s = s + 2;
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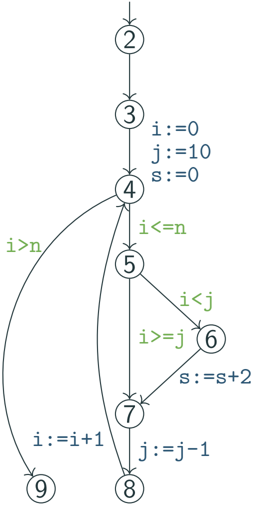


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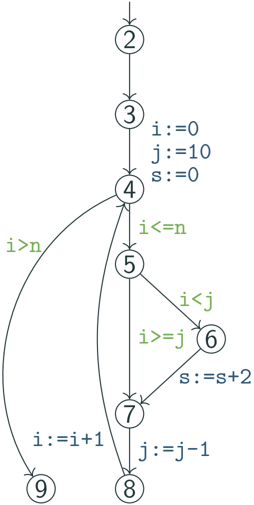

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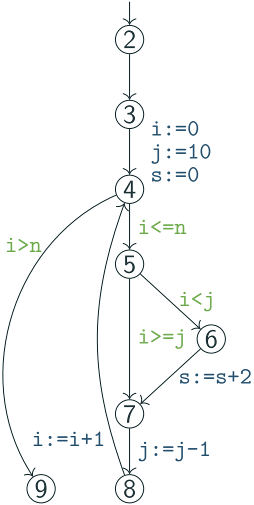

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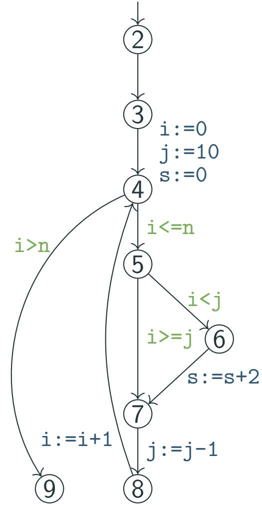

## Software Verification Competition (SV-COMP)

- annual competition
  `https://sv-comp.sosy-lab.org/2018/`
- industrial (and crafted) benchmarks
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- explicit assertions

# Outline

- Summary of Last Week
- Bounded Model Checking for Verification
- Quantifiers for SMT
  - Skolemization

# Applications of Quantifiers in SMT

## Example (Homework)

Imagine a village of monkeys where each monkey owns at least two bananas. As the monkeys are well-organised, each tree contains exactly three monkeys. Monkeys are also very friendly, so every monkey has a partner.

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  $\forall x\ y\ z.\ \mathsf{inv}(x) \cdot x = 0 \wedge 0 \cdot x = x \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z$

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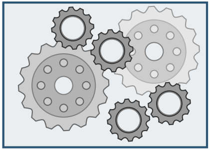
  $\forall input.\ \exists output.\ \mathrm{F}(input, output)$
- planning
  $\exists plan.\ \forall time.\ \mathrm{spec}(plan, time)$

# SMT Solving with Quantifiers

## SMT solver

$\varphi$ $\longrightarrow$

$f(a) = a \vee (f(b) \neq a \wedge a = b)$



SAT ($v$) $f(a) = a$

UNSAT

## Decision Properties

- SMT solvers can decide propositional logic + LIA/LRA/EUF/BV/...

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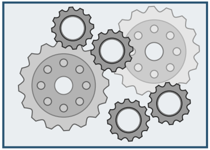

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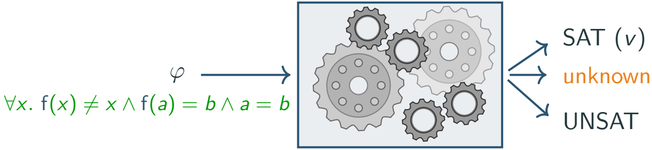


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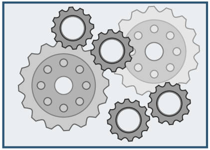

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first-order logic is undecidable!

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# Skolemization

Thoralf Skolem

## Getting rid of $\exists$ quantifiers

- replace $\exists x.\ \mathrm{P}(x)$ by $\mathrm{P}(\mathrm{a})$

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can consider formulas of shape $\forall x_1, \dots, x_n\ \varphi[x_1, \dots, x_n]$

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Herbrand instance of Skolem formula $\forall x_1, \ldots, x_n \; \varphi[x_1, \ldots, x_n]$ is $\varphi[t_1, \ldots, t_n]$ where $t_i$ is term over signature of $\varphi$

**Definition**

set of function symbols and constants

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## Caveats

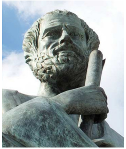
- at least one constant required per sort
- holds for pure first order logic, not necessarily in presence of theories

## Example: Is this syllogism correct?

All humans are mortal.
All Greeks are humans.

---

So all Greeks are mortal.



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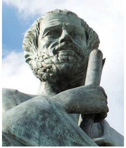

Aristotle

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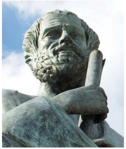

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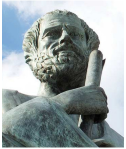

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cannot be answered by SMT solver

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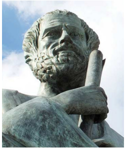
| All Greeks are humans. | $\forall x.\ G(x) \longrightarrow H(x)$ |
| So all Greeks are mortal. | $\forall x.\ G(x) \longrightarrow M(x)$ |



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- check unsatisfiability of

$$\forall x.\ H(x) \longrightarrow M(x), \quad \forall x.\ G(x) \longrightarrow H(x), \quad \exists x.\ G(x) \wedge \neg M(x)$$

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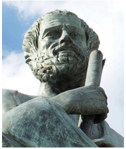
| All humans are mortal. | $\forall x.\ H(x) \longrightarrow M(x)$ |
| All Greeks are humans. | $\forall x.\ G(x) \longrightarrow H(x)$ |
| So all Greeks are mortal. | $\forall x.\ G(x) \longrightarrow M(x)$ |



Aristotle

- translate to first-order logic
- check validity of

$$((\forall x.\ H(x) \longrightarrow M(x)) \wedge (\forall x.\ G(x) \longrightarrow H(x))) \longrightarrow (\forall x.\ G(x) \longrightarrow M(x))$$

- check unsatisfiability of

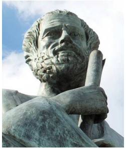
$$\forall x.\ H(x) \longrightarrow M(x), \quad \forall x.\ G(x) \longrightarrow H(x), \quad \exists x.\ G(x) \wedge \neg M(x)$$

- skolemize

$$\forall x.\ H(x) \longrightarrow M(x), \quad \forall x.\ G(x) \longrightarrow H(x), \quad G(a) \wedge \neg M(a)$$

## Example: Is this syllogism correct?

$$\frac{\begin{array}{l}\text{All humans are mortal.}\\ \text{All Greeks are humans.}\end{array}}{\text{So all Greeks are mortal.}} \qquad \frac{\begin{array}{l}\forall x.\ H(x) \longrightarrow M(x)\\ \forall x.\ G(x) \longrightarrow H(x)\end{array}}{\forall x.\ G(x) \longrightarrow M(x)}$$



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$$\forall x.\ H(x) \longrightarrow M(x), \quad \forall x.\ G(x) \longrightarrow H(x), \quad G(a) \wedge \neg M(a)$$

- already unsatisfiable when replacing quantified formulas by Herbrand instances

$$H(a) \longrightarrow M(a), \quad G(a) \longrightarrow H(a), \quad G(a) \wedge \neg M(a)$$

## Example: Is this syllogism correct?

| | |
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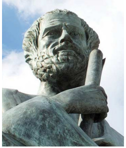

Aristotle

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- check validity of

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$$\forall x.\ H(x) \longrightarrow M(x), \quad \forall x.\ G(x) \longrightarrow H(x), \quad \exists x.\ G(x) \wedge \neg M(x)$$

- skolemize

$$\forall x.\ H(x) \longrightarrow M(x),$$

when adding right Herbrand instances
unsatisfiability can be detected by SMT solver

- already unsatisfiable when replacing quantified formulas by Herbrand instances

$$H(a) \longrightarrow M(a), \quad G(a) \longrightarrow H(a), \quad G(a) \wedge \neg M(a)$$

# Bibliography

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