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SAT and SMT Solving

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Outline

- Summary of Last Week
- Bounded Model Checking for Verification
- Quantifiers for SMT

Definitions

- theory consists of
  - signature Σ: set of function and predicate symbols
  - axioms T: set of sentences in first-order logic in which only function and predicate symbols of Σ appear
- theory is stably infinite if every satisfiable quantifier-free formula has model with infinite carrier set
- theory T is convex if $F \vDash_T \bigvee_{i=1}^{n} u_i = v_i$ implies $F \vDash_T u_i = v_i$ for some $1 \leqslant i \leqslant n$ $\forall$ quantifier-free conjunction F and variables u_i, v_i

Definition

theory combination $T_1 \oplus T_2$: signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$

Assumptions

two stably infinite theories T_1, T_2 over signatures Σ_1, Σ_2 such that

- $\Sigma_1 \cap \Sigma_2 = \{=\}$
- T_i-satisfiability of quantifier-free Σ_i-formulas is decidable for $i \in \{1, 2\}$

Nelson-Oppen Method: Nondeterministic Version

Input — quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$

Output — satisfiable or unsatisfiable

1. purification

$$\varphi \;\approx\; \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - V is set of shared variables in φ_1 and φ_2
   - guess equivalence relation E on V
   - arrangement $\alpha(V, E)$ is formula

$$\bigwedge_{x\,E\,y} x = y \quad \wedge \quad \bigwedge_{\neg(x\,E\,y)} x \neq y$$

   - if $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

Nelson-Oppen Method: Deterministic Version

Input quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories T_1 and T_2

Output satisfiable or unsatisfiable

1. purification $\varphi \approx \varphi_1 \wedge \varphi_2$ for Σ_1-formula φ_1 and Σ_2-formula φ_2
2. V: set of shared variables in φ_1 and φ_2

   E: already discovered equalities between variables in V
3. test satisfiability of $\varphi_1 \wedge E$ (and add implied equations)
   - if $\varphi_1 \wedge E$ is T_1-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
4. test satisfiability of $\varphi_2 \wedge E$ (and add implied equations)
   - if $\varphi_2 \wedge E$ is T_2-unsatisfiable then return unsatisfiable
   - else add new implied equalities to E
5. if E has been extended in steps 3 or 4 then go to step 2

   else return satisfiable

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Disastrous Software Bugs

Ariane 5 Flight 501 (1996)

- destroyed 37 seconds after launch
- software for Ariane 4 for was reused
- software error: data conversion from 64-bit floating point to 16-bit integer caused arithmetic overflow
- cost: 370 million $

http://en.wikipedia.org/wiki/Ariane_5_Flight_501

Mars Exploration Rover “Spirit” (2004)

- landed on January 4
- stopped communicating on January 21
- software error: stuck in reboot loop
- reboot failed because of flash memory failure, ultimate problem: too many files

http://en.wikipedia.org/wiki/Spirit_(rover)

Heathrow Terminal 5 Opening (2008)

- baggage system collapsed on opening day
- 42,000 bags not shipped with their owners, 500 flights cancelled
- software was tested but did not work properly with real-world load
- cost 50 million £

http://www.zdnet.com/article/it-failure-at-heathrow-t5-what-really-happened

Trading Glitch at Knight Capital (2012)

- bug in trading software resulted in 45 minutes of uncontrolled buys
- company did 11% of US trading that year
- software was run in invalid configuration
- 440 million $ lost

http://en.wikipedia.org/wiki/Knight_Capital_Group

Death in Self-Driving Car Crash (2018)

- person died in accident with Uber's self-driving car
- victim was wrongly classified by software as non-obstacle

http://www.siliconrepublic.com/companies/uber-bug-crash

Software is Ubiquitous in Critical Systems

transport, energy, medicine, communication, finance, embedded systems, ...

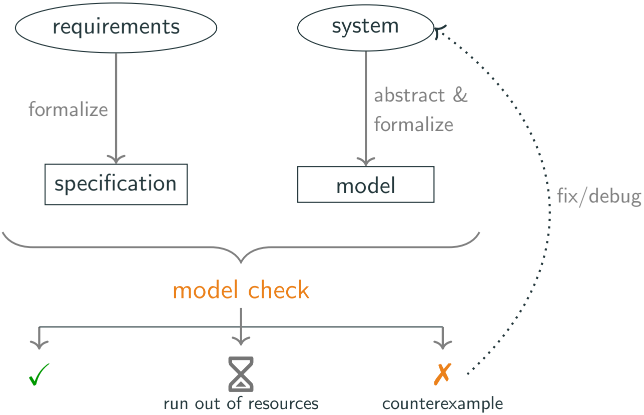
How to Ensure Correctness of Software?

- testing
  - \+ cheap, simple
  - – checks desired result only for given set of testcases
- verification
  - \+ can prove automatically that system meets specification, i.e., desired output is delivered for all inputs
  - – more costly

Model Checking

- widely used verification approach to
  - find bugs in software and hardware
  - prove correctness of models
- Turing Award 2007 for Clarke, Emerson, and Sifakis
- bounded model checking can be reduced to SAT/SMT

Model Checking: Workflow

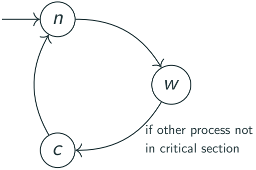


Model Checking Example: Mutex (1)

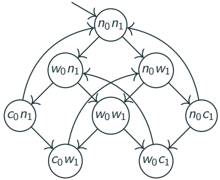
- concurrent processes P_0, P_1 share some resource, access controlled by mutex
- program run by P_0, P_1 matches pattern

```
# non-critical section
while (other process critical) :
  wait ()
# critical section
# non-critical section
```

- process can be abstracted to model $\mathcal{M} = \langle S, R \rangle$
  with states $S = \{n, w, c\}$ and transitions R:



Model Checking Example: Mutex (2)



- obtain model for 2 processes by product construction: write $s_0 s_1$ for P_0 being in state s_0 and P_1 in state s_1

- desired properties:

  safe: only one process is in its critical section at any time

  live: whenever any process wants to enter its critical section, it will eventually be permitted to do so

  non-blocking: a process can always request to enter its critical section

- how to formalize desired properties? temporal logic, e.g. LTL or CTL

  safe: $\mathsf{G}\ \neg(c_0 \wedge c_1)$ ✓ as $c_0 c_1$ unreachable

  live: $\mathsf{G}\ (w_0 \to \mathsf{F}\ c_0)$ ✗ e.g. $w_0 n_1 \to w_0 w_1 \to w_0 c_1 \to w_0 n_1 \to \cdots$

  non-blocking: $\mathsf{AG}\ (n_0 \to \mathsf{EX}\ w_0)$ ✓

Observation

model checking is feasible for this example because state space is finite and small

Common Kinds of Properties

Safety property

- "bad things don't happen"
- expressed as $\mathsf{G}\ \psi$, for some ψ without temporal operators
- violated within finite number of steps

Liveness property

- "good things happen eventually"
- expressed as $\mathsf{G}\ (\psi \to \mathsf{F}\chi)$, for some ψ, χ without temporal operators

Example

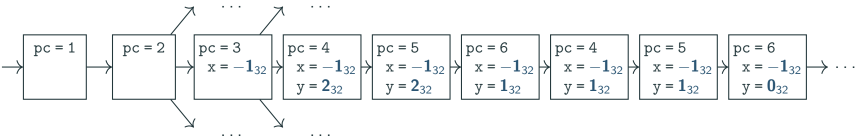
- safety properties
  - program never reaches an error state $\mathsf{G}(\neg \mathit{error})$
  - programm does not violate access permissions $\mathsf{G}(\neg \mathit{violation})$
  - program never uses more than 1GB of RAM $\mathsf{G}(\mathit{mem} < 1\mathit{GB})$
- liveness properties
  - every task is eventually processed $\mathsf{G}(\mathit{task\ created} \to \mathsf{F}\mathit{processed})$
  - the database is eventually consistent $\mathsf{G}(\mathit{change\ DB} \to \mathsf{F}\mathit{consistent})$
  - if user inputs *a*, program eventually does *b* $\mathsf{G}(a \to \mathsf{F}b)$

Example: Can This Program Cause An Overflow?

```
1 void main() {
2   int x = -1;
3   int y = input();
4   while (y<100) {
5     y = y+x;
6   }
7 }
```

addition $x + y$ in line 5 does not over/underflow

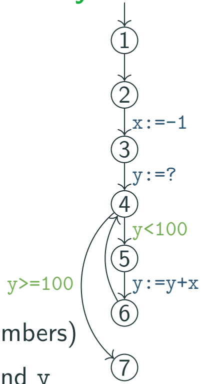
- model checking problem:
  - state consists of values for (x, y) + value of program counter pc
  - safety property $\mathsf{G}\ ((x > \mathbf{0}_{32} \wedge x{+}y < y) \vee (x \leqslant \mathbf{0}_{32} \wedge x{+}y > y))$
- (part of) model:



- but state space is very large: $(2^{32})^2 \cdot 7$ for bit width 32
- cannot check all possible values

Example: Can This Program Cause An Overflow? Next try.

```
1 void main() {
2   int x = -1;
3   int y = input();
4   while (y<100) {
5     y = y+x;
6   }
7 }
```

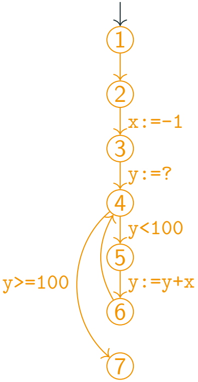


- construct program graph G
- $\{1, \ldots, 7\}$ are possible values of program counter (line numbers)
- state is tuple $\langle \mathrm{pc}, x, y \rangle$ of values of program counter, `x`, and `y`
- state of form $\langle 1, \ldots, \ldots \rangle$ is initial state
- examples of state transitions according to G:
  - $\langle 4, -\mathbf{1}_{32}, \mathbf{10}_{32} \rangle \to \langle 5, -\mathbf{1}_{32}, \mathbf{10}_{32} \rangle$ is possible
  - $\langle 4, -\mathbf{1}_{32}, \mathbf{101}_{32} \rangle \to \langle 7, -\mathbf{1}_{32}, \mathbf{101}_{32} \rangle$ is possible
  - $\langle 4, \mathbf{10}_{32}, \mathbf{101}_{32} \rangle \to \langle 5, \mathbf{10}_{32}, \mathbf{101}_{32} \rangle$ is not possible
  - $\langle 4, -\mathbf{1}_{32}, \mathbf{1}_{32} \rangle \to \langle 5, -\mathbf{1}_{32}, \mathbf{2}_{32} \rangle$ is not possible

Idea

consider symbolic program executions with bounded length,
try to solve with SMT solver

Example: Can This Program Cause An Overflow?



1 define predicates

- $I(\langle pc, x, y\rangle) = (pc = 1)$ to characterize initial state
- to characterize possible state transitions:

$$
\begin{aligned}
&T(\langle pc, x, y\rangle, \langle pc', x', y'\rangle) = \\
&\quad (pc = 1 \wedge pc' = 2) \vee (pc = 2 \wedge pc' = 3 \wedge x' = -1) \vee \\
&\quad (pc = 3 \wedge pc' = 4 \wedge x = x') \vee \\
&\quad (pc = 4 \wedge pc' = 5 \wedge y < 100 \wedge x = x' \wedge y = y') \vee \\
&\quad (pc = 5 \wedge pc' = 6 \wedge y' = y + x \wedge x = x') \vee \\
&\quad (pc = 4 \wedge pc' = 7 \wedge y \geqslant 100 \wedge x = x' \wedge y = y') \vee \\
&\quad (pc = 6 \wedge pc' = 4 \wedge x = x' \wedge y = y')
\end{aligned}
$$

- $P(\langle pc, x, y\rangle) = (pc = 5) \wedge ((x > \mathbf{0}_{32} \wedge x + y \leqslant y) \vee (x \leqslant \mathbf{0}_{32} \wedge (y + x > y)))$

2 for states $s_0, \ldots, s_k$ formula φ_k expresses overflow occurring within k steps:

$$\varphi_k = I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} P(s_i)$$

3 if φ_k satisfiable then overflow can occur within k steps, e.g. for $k = 5$

Bounded Model Checking

- find counterexamples to desired property of transition system (bugs)
- counterexamples are bounded in size

Definition (Transition System)

transition system $\mathcal{T} = (S, \rightarrow, S_0, L)$ where

- S is set of states
- $\rightarrow \subseteq S \times S$ is transition relation
- $S_0 \subseteq S$ is set of initial states
- A is a set of propositional atoms
- $L : S \rightarrow 2^A$ is labeling function associating state with subset of A

Remark

S and A may be (countably) infinite

Bounded Model Checking: Safety Properties

Idea

given transition system and property $\mathsf{G}\ \psi$, look for counterexamples in $\leqslant k$ steps

$$\to s_0 \to s_1 \to s_2 \to \cdots \to s_k$$

$$\neg\psi \ \vee\ \neg\psi \ \vee\ \neg\psi \ \vee\ \cdots \ \vee\ \neg\psi$$

SAT/SMT Encoding

given transition system $\mathcal{T}$ and safety property $\mathsf{G}\ \psi$

- use encoding $\langle s \rangle$ of state $s \in S$ by set of SAT/SMT variables
- use predicates
  - I for initial states such that use $I(\langle s \rangle)$ is true iff $s \in S_0$
  - T for transitions such that $T(\langle s \rangle, \langle s' \rangle)$ is true iff $s \to s'$ in $\mathcal{T}$
  - P such that $P(\langle s \rangle)$ is true iff ψ holds in s
- use different fresh variables for $k+1$ states $\langle s_0 \rangle, \ldots, \langle s_k \rangle$
- check satisfiability of

$$I(\langle s_0 \rangle) \wedge \bigwedge_{i=0}^{k-1} T(\langle s_i \rangle, \langle s_{i+1} \rangle) \wedge \bigvee_{i=0}^{k} \neg P(\langle s_i \rangle)$$

Bounded Model Checking: Liveness Properties

Idea

- counterexample to liveness property G $(\psi \to \mathsf{F}\chi)$ requires infinite path
- look for counterexamples in $\leqslant k$ steps of lasso shape:

$$\begin{tikzcd}
\arrow[r] & s_0 \arrow[r] & \cdots \arrow[r] & s_i \arrow[r] & \cdots \arrow[r] & s_l \arrow[r] & \cdots \arrow[r] & s_k \arrow[ll, bend right] \\
& & & \begin{matrix}\neg\chi \\ \psi\end{matrix} & \wedge \ \cdots \ \wedge & \neg\chi & \wedge \ \cdots \ \wedge & \neg\chi
\end{tikzcd}$$

SAT/SMT Encoding

given transition system $\mathcal{T}$ and liveness property G $(\psi \to \mathsf{F}\chi)$

- use encoding of states, predicates I and T as for safety properties
- predicate P such that $P(\langle s\rangle)$ is true iff ψ holds in s
- predicate C such that $C(\langle s\rangle)$ is true iff χ holds in s
- check satisfiability of

$$I(\langle s_0\rangle) \wedge \bigwedge_{i=0}^{k-1} T(\langle s_i\rangle, \langle s_{i+1}\rangle) \wedge \bigvee_{i=0}^{k} \left(P(\langle s_i\rangle) \wedge \bigwedge_{j=i}^{k} \neg C(\langle s_j\rangle) \wedge \bigvee_{l=i}^{k} T(\langle s_k\rangle, \langle s_l\rangle) \right)$$

Transition System $\mathcal{T}(P)$ from Program P

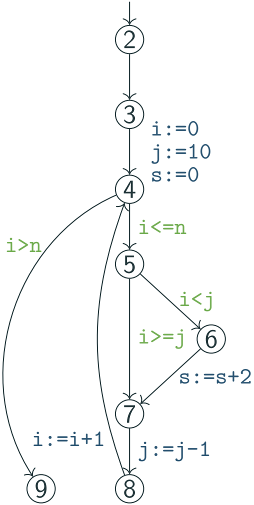
- state $\langle pc, v_0, \ldots, v_n \rangle$ consists of
  - value for program counter pc, i.e. line number in P
  - assignment for variables in scope $\mathrm{v}_0, \ldots, \mathrm{v}_n$
- there is step $s \to s'$ for $s = \langle pc, v_0, \ldots, v_n \rangle$ and $s' = \langle pc', v_0', \ldots, v_n' \rangle$ iff P admits step from s to s'
- S_0 consists of initial program states
- atom set A consists of all propositional formulas over $pc, v_0, \ldots, v_n$
- labeling $L(s)$ is set of all atoms A which hold in $s = \langle pc, v_0, \ldots, v_n \rangle$

Program Graph

- nodes are line numbers
- edge from line l to line l' if program counter can go from line l to l'
- two kinds of edge labels:
  - conditions for program counter to take this path
  - assignments of updated variables
- program graph is useful to derive encoding of $\mathcal{T}(P)$

Checking an Explicit Assertion

```
1  int n;
2  int main() {
3    int i=0, j=10, s=0;
4    for(i=0; i<=n; i++) {
5      if (i<j)
6        s = s + 2;
7      j--;
8    }
9    assert(s==n*2 || s == 0);
10 }
```



- construct program graph
- states are of form $\langle \mathrm{pc}, i, j, n, s \rangle$
- safety property to check is $\mathrm{G}\ (\mathrm{pc} = 9 \rightarrow (s = 2n \lor s = 0))$
- see `verification.py`

Software Verification Competition (SV-COMP)

- annual competition
  `https://sv-comp.sosy-lab.org/2018/`
- industrial (and crafted) benchmarks
  `https://github.com/sosy-lab/sv-benchmarks`
- many tools use SMT solvers

Common Safety Properties

- no overflow in addition: $(\mathtt{x} > 0 \land \mathtt{x} + \mathtt{y} \geqslant \mathtt{y}) \lor (\mathtt{x} \leqslant 0 \land \mathtt{x} + \mathtt{y} \leqslant \mathtt{y})$
- array accesses in bounds: $0 \leqslant \mathtt{i} < \mathit{size}(\mathtt{a})$ for all accesses $\mathtt{a}[\mathtt{i}]$
- memory safety: set predicate $\mathit{ok}(\mathit{addr})$ when memory allocated, check $\mathit{ok}(\mathtt{p})$ for every dereference $*\mathtt{p}$
- explicit assertions

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    - Skolemization

Applications of Quantifiers in SMT

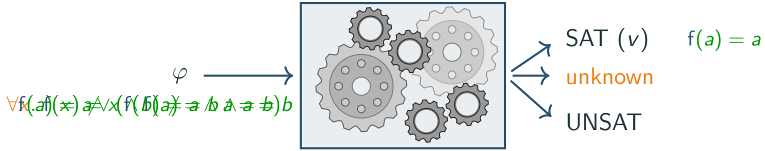
Example (Homework)

Imagine a village of monkeys where each monkey owns at least two bananas. As the monkeys are well-organised, each tree contains exactly three monkeys. Monkeys are also very friendly, so every monkey has a partner.

More important applications

- automated theorem proving
  $\forall x\ y\ z.\ \mathsf{inv}(x) \cdot x = 0 \wedge 0 \cdot x = x \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- software verification
  $\forall x.\ \mathsf{pre}(x) \longrightarrow \mathsf{post}(x)$
- function synthesis
  $\forall \mathit{input}.\ \exists \mathit{output}.\ \mathsf{F}(\mathit{input}, \mathit{output})$
- planning
  $\exists \mathit{plan}.\ \forall \mathit{time}.\ \mathsf{spec}(\mathit{plan}, \mathit{time})$

SMT solver



Decision Properties

first-order logic is undecidable!

- SMT solvers can decide propositional logic + LIA/LRA/EUF/BV/...
- many SMT solvers also have support for quantifiers,
  but have in general no decision procedure for theories + quantifiers

Skolemization

name witness for existential quantifier

Thoralf Skolem

Getting rid of $\exists$ quantifiers

- replace $\exists x.\ \mathsf{P}(x)$ by $\mathsf{P}(\mathsf{a})$
- replace $\forall y\ \exists x.\ \mathsf{P}(x)$ by $\forall y\ \mathsf{P}(\mathsf{f}(y))$
- replace $\forall z\ \forall y\ \exists x.\ \mathsf{R}(x)$ by $\forall z\ \forall y\ \mathsf{R}(\mathsf{f}(y,z))$

Definitions

- φ is in prenex form if $\varphi = Q_1 x_1 \dots Q_n x_n\ \psi$ for ψ quantifier-free and $Q_i \in \{\forall, \exists\}$
- φ is in Skolem form if in prenex form without existential quantifier

Skolemization

1. bring formula into prenex form
2. replace $\forall x_1, \dots, x_k \exists y\ \psi[y]$ by $\forall x_1, \dots, x_k\ \psi[\mathsf{f}(x_1, \dots, x_k)]$ for fresh f

   until no existential quantifiers left

can consider formulas of shape $\forall x_1, \dots, x_n\ \varphi[x_1, \dots, x_n]$

Theorem

if φ' is skolemization of φ then φ and φ' are equisatisfiable

Definition

set of function symbols and constants

Herbrand instance of Skolem formula $\forall x_1, \ldots, x_n\ \varphi[x_1, \ldots, x_n]$ is $\varphi[t_1, \ldots, t_n]$
where t_i is term over signature of φ

Remark

Herbrand instances are ground formulas, i.e., without (quantified) variables

Theorem (Herbrand)

Skolem formula φ is unsatisfiable $\Longleftrightarrow$
there exists finite unsatisfiable set of Herbrand instances of φ

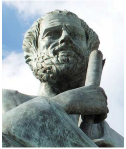
Jacques Herbrand

Caveats

- at least one constant required per sort
- holds for pure first order logic, not necessarily in presence of theories

Example: Is this syllogism correct?

All humans are mortal.	$\forall x.\ H(x) \longrightarrow M(x)$
All Greeks are humans.	$\forall x.\ G(x) \longrightarrow H(x)$
So all Greeks are mortal.	$\forall x.\ G(x) \longrightarrow M(x)$


Aristotle

- translate to first-order logic
- check validity of

cannot be answered by SMT solver

$$((\forall x.\ H(x) \longrightarrow M(x)) \wedge (\forall x.\ G(x) \longrightarrow H(x))) \longrightarrow (\forall x.\ G(x) \longrightarrow M(x))$$

- check unsatisfiability of

$$\forall x.\ H(x) \longrightarrow M(x), \quad \forall x.\ G(x) \longrightarrow H(x), \quad \exists x.\ G(x) \wedge \neg M(x)$$

- skolemize

$$\forall x.\ H(x) \longrightarrow M(x),$$

when adding right Herbrand instances
unsatisfiability can be detected by SMT solver

- already unsatisfiable when replacing quantified formulas by Herbrand instances

$$H(a) \longrightarrow M(a), \quad G(a) \longrightarrow H(a), \quad G(a) \wedge \neg M(a)$$

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