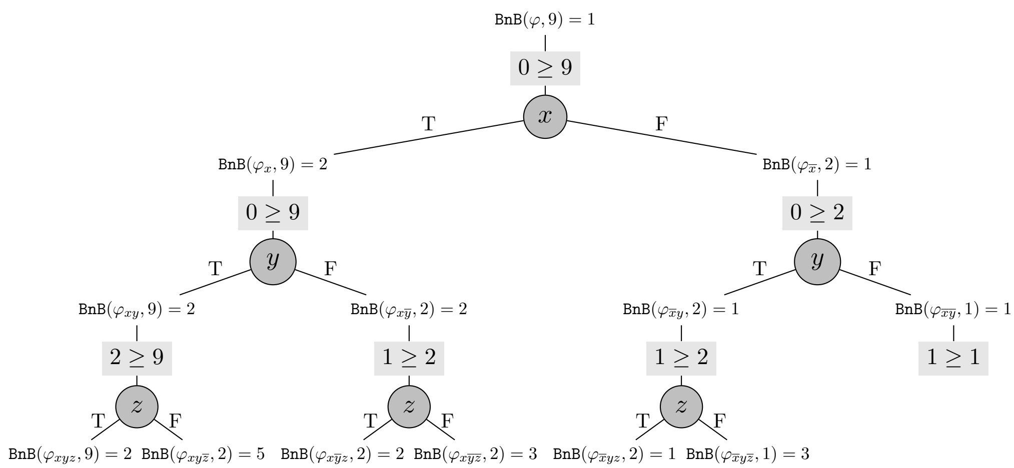
SAT and SMT Solving | WS 2022 | LVA 703147

Selected Solutions 3 | November 4, 2022

$\boxed{1}$ The following computation yields $\texttt{BnB}(\varphi, 9) = 1$, hence $\text{maxSAT}(\varphi) = 8$.



$$
\begin{aligned}
\texttt{simp}(\varphi_x) &= \neg y,\ (\mathrm{T} \vee y),\ (\neg\mathrm{T} \vee z),\ (\neg\mathrm{T} \vee y),\ (\neg\mathrm{T} \vee \neg y),\ (\mathrm{T} \vee z),\ (\neg\mathrm{T} \vee \neg y \vee z),\\
&\quad (y \vee \neg z),\ z\\
&= \neg y,\ z,\ y,\ \neg y,\ (\neg y \vee z),\ (y \vee \neg z),\ z\\
\texttt{simp}(\varphi_{xy}) &= \neg\mathrm{T},\ z,\ \mathrm{T},\ \neg\mathrm{T},\ (\neg\mathrm{T} \vee z),\ (\mathrm{T} \vee \neg z),\ z = \Box,\ z,\ \ \Box,\ z,\ z\\
\texttt{simp}(\varphi_{xyz}) &= \Box,\ \Box\\
\texttt{simp}(\varphi_{xy\overline{z}}) &= \Box,\ \Box,\ \Box,\ \Box,\ \Box\\
\texttt{simp}(\varphi_{x\overline{y}}) &= \neg\mathrm{F},\ z,\ \mathrm{F},\ \neg\mathrm{F},\ (\neg\mathrm{F} \vee z),\ (\mathrm{F} \vee \neg z),\ z = z,\ \Box,\ \neg z,\ z\\
\texttt{simp}(\varphi_{x\overline{y}z}) &= \Box,\ \Box\\
\texttt{simp}(\varphi_{x\overline{y}\overline{z}}) &= \Box,\ \Box,\ \Box\\
\texttt{simp}(\varphi_{\overline{x}}) &= \neg y,\ (\mathrm{F} \vee y),\ (\neg\mathrm{F} \vee z),\ (\neg\mathrm{F} \vee y),\ (\neg\mathrm{F} \vee \neg y),\ (\mathrm{F} \vee z),\ (\neg\mathrm{F} \vee \neg y \vee z),\\
&\quad (y \vee \neg z),\ z\\
&= \neg y,\ y,\ z,\ (y \vee \neg z),\ z\\
\texttt{simp}(\varphi_{\overline{x}y}) &= \Box,\ z,\ z\\
\texttt{simp}(\varphi_{\overline{x}yz}) &= \Box\\
\texttt{simp}(\varphi_{\overline{x}y\overline{z}}) &= \Box,\ \Box,\ \Box\\
\texttt{simp}(\varphi_{\overline{x}\overline{y}}) &= \Box,\ z,\ \neg z,\ z
\end{aligned}
$$

$\boxed{2}$ See the file `minUnsat.py`.

$\boxed{4}$ Let φ be a CNF formula given as a list of clauses; where for the sake of the induction proof below, a clause is a list that contains variables, T or F. We prove by induction on the number of variables in φ that for any $k \in \mathbb{N}$, $\texttt{BnB}(\varphi, k)$ returns either minUNSAT(φ) or min(minUNSAT(φ), k).

This suffices to show that $\texttt{BnB}(\varphi, |\varphi|)$ returns minUNSAT(φ), because minUNSAT(φ) $\leq |\varphi|$ implies that minUNSAT(φ) = min(minUNSAT(φ), $|\varphi|$). Below we will use the fact that $\texttt{simp}(\psi) \equiv \psi$ for any formula ψ, which is easy to show, so that minUNSAT($\texttt{simp}(\psi)$) = minUNSAT(ψ).

For the base case, suppose φ has no variables. Then $\texttt{simp}(\varphi)$ can contain only empty clauses, and the number of empty clauses m in $\texttt{simp}(\varphi)$ is the number of clauses falsified by every assigment, so $m =$ minUNSAT(φ). Thus the first case of the claim is satisfied.

Now let φ contain at least one variable. If $\texttt{simp}(\varphi)$ contains only empty clauses, we can reason as in the base case that $\texttt{BnB}(\varphi, k)$ returns minUNSAT(φ). Otherwise, let m be the number of empty clauses in $\texttt{simp}(\varphi)$. If $m \geq k$, $\texttt{BnB}(\varphi, k)$ returns k by definition. No valuation can satisfy empty clauses, so in this case we have min(minUNSAT(φ), k) = k, and the claim holds.

Otherwise, $m < k$. Let x be the selected variable. The formulas $\varphi_{\overline{x}}$ and φ_x have fewer variables than φ, so for both $\varphi' \in \{\varphi_{\overline{x}}, \varphi_x\}$, by the induction hypothesis either $\texttt{BnB}(\varphi', n) =$ min(minUNSAT(φ'), n) or $\texttt{BnB}(\varphi', n) =$ minUNSAT(φ') holds for all n. Therefore, for $k' := \texttt{BnB}(\varphi_x, k)$, we have $k' =$ min(minUNSAT(φ_x), k) or $k' =$ minUNSAT(φ_x). Similarly, for $k'' := \texttt{BnB}(\varphi_{\overline{x}}, k')$ we have $k'' =$ min(minUNSAT($\varphi_{\overline{x}}$), k') or $k'' =$ minUNSAT($\varphi_{\overline{x}}$). By definition (last two lines), $\texttt{BnB}(\varphi, k)$ returns the minimum of k, k', and k'', that is, it returns min(k, minUNSAT(φ_x), minUNSAT($\varphi_{\overline{x}}$). By definition, $N :=$ minUNSAT(φ) is the minimum number of clauses falsified by an assignment. Such an assignment must assign T or F to x. In the former case, we have $N =$ minUNSAT(φ_x) and in the latter $N =$ minUNSAT($\varphi_{\overline{x}}$), but in either case N is the minimum of the two. So by the observation above, $\texttt{BnB}(\varphi, k)$ returns indeed min(k, minUNSAT(φ_x), minUNSAT($\varphi_{\overline{x}}$) = min(k, minUNSAT(φ)), so the claim holds.[1]

[1] Bugs in an earlier version fixed thanks to James Fox.