Selected Solutions 11 January 20, 2023

1 a) The given formula (call it φ_1) can be purified to

$$\psi_1 := z = 0 \land y \leq x \land x \leq y + z \land v = 1 \land w = 2$$
$$\psi_2 := \mathsf{f}(y) = v \land \mathsf{f}(z) = w \land \mathsf{f}(y) = \mathsf{f}(z)$$

One can use the deterministic Nelson-Oppen procedure since both LRA and EUF are convex. We initialize $E := \varnothing$, and proceed as follows:

- In EUF, $\psi_2 \land E$ is satisfiable and implies $v = w$, so we update $E := \{v = w\}$.
- In LRA, $\psi_1 \land E$ is unsatisfiable, so φ_1 is unsatisfiable.

b) The given formula (call it φ_2) is easy to purify by splitting literals, into

$$\chi_1 := x = y + 1 \land y \leq z \land x \geq z + 1$$
$$\chi_2 := \mathsf{f}(y) = \mathsf{a} \land \mathsf{f}(z) = \mathsf{b}$$

The shared variables are $\{y, z\}$. Using the nondeterministic Nelson-Oppen procedure, we guess the equivalence relation $\{\{z, y\}\}$, with arrangement $\alpha := (y = z)$. Since $\chi_1 \land \alpha$ is satisfiable in LRA and $\chi_2 \land \alpha$ is satisfiable in EUF, φ_2 is satisfiable.

4 Consider the formula $\varphi := \mathsf{f}(z) = \mathsf{g}(z)$ for a variable z, which is satisfiable in T. For instance, φ has the model $\mathcal{M}$ whose domain is a singleton set $\{a\}$, and $\mathsf{f}_{\mathcal{M}}(x, y) = x$, $\mathsf{g}_{\mathcal{M}}(x) = x$, and $\mathsf{h}_{\mathcal{M}}(x) = x$.

Now we consider an arbitrary model of φ. Any such model has a domain D and assigns some value to z. In combination with the axioms, φ implies

$$z = \mathsf{f}(\mathsf{g}(x), \mathsf{h}(z))) = \mathsf{f}(\mathsf{g}(x), \mathsf{g}(z))) = x$$

This means that in any model of φ, all terms are equal to the value assigned to z, so all models have a domain whose carrier is a singleton set. In particular, T does not have a model with an infinite carrier.

Thus T is not stable infinite.