1st EXAM – SOLUTIONS  January 29, 2024

1 (a) The set $A = \{(a^n b)^m \mid n > m > 0\}$ is not regular. Consider the strings $x_i = a^i b$ for $i \geqslant 1$. If $i > j > 0$ then $x_i(a^i b)^{j-1} \in A$ and $x_j(a^i b)^{j-1} \notin A$ and thus $x_i \not\equiv_A x_j$. Hence $\equiv_A$ has infinite index and thus the non-regularity of $A$ follows from the Myhill–Nerode theorem.

(b) The set $B = \{x^n \mid \#a(x) \neq \#b(x) \text{ and } n > 0\}$ is not regular. Consider the strings $x_i = a^i$ for $i \geqslant 1$. If $i \neq j$ then $x_i b^i \notin B$ and $x_j b^i \in B$ and thus $x_i \not\equiv_B x_j$. Hence $\equiv_B$ has infinite index and thus the non-regularity of $B$ follows from the Myhill–Nerode theorem.

2 (a) Using the construction from the lecture we obtain

$$\varphi = \exists X_1.\, \exists X_2.\, \exists X_3.\, \exists \ell. \neg P_a(\ell) \wedge \neg P_b(\ell) \wedge \Big(\forall x.\, \neg P_a(x) \wedge \neg P_b(x) \rightarrow \ell \leqslant x\Big) \wedge \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4$$

with

$$\begin{aligned}
\psi_1 &= X_1(0)\\
\psi_2 &= \forall x.\, x \leqslant \ell \rightarrow \big(X_1(x) \vee X_2(x) \vee X_3(x)\big) \wedge \neg\big(X_1(x) \wedge X_2(x)\big) \wedge \neg\big(X_1(x) \wedge X_3(x)\big) \wedge\\
&\qquad \neg\big(X_2(x) \wedge X_3(x)\big)\\
\psi_3 &= \forall x.\, x < \ell \rightarrow (X_1(x) \wedge P_a(x) \wedge \exists y.\, y = x+1 \wedge X_2(y)) \vee\\
&\qquad (X_1(x) \wedge P_b(x) \wedge \exists y.\, y = x+1 \wedge X_1(y)) \vee\\
&\qquad (X_2(x) \wedge P_a(x) \wedge \exists y.\, y = x+1 \wedge X_2(y)) \vee\\
&\qquad (X_2(x) \wedge P_b(x) \wedge \exists y.\, y = x+1 \wedge X_3(y)) \vee\\
&\qquad (X_3(x) \wedge P_a(x) \wedge \exists y.\, y = x+1 \wedge X_2(y)) \vee\\
&\qquad (X_3(x) \wedge P_b(x) \wedge \exists y.\, y = x+1 \wedge X_3(y))\\
\psi_4 &= X_1(\ell) \vee X_3(\ell)
\end{aligned}$$

(b) First consider the subformula $\psi = y < x \wedge X(y)$ with $\mathsf{FV}(\psi) = (x, y, X)$. We have

$$L_a(y < x) = \tbinom{0}{0}^* \tbinom{0}{1} \tbinom{0}{0}^* \tbinom{1}{0} \tbinom{0}{0}^*$$

$$L_a(X(y)) = \left[\tbinom{0}{0} + \tbinom{0}{1}\right]^* \tbinom{1}{1} \left[\tbinom{0}{0} + \tbinom{0}{1}\right]^*$$

and compute

$$L_1 = \mathsf{drop}_3^{-1}(L_a(y < x)) = \begin{pmatrix}0\\0\\*\end{pmatrix}^* \begin{pmatrix}1\\0\\*\end{pmatrix} \begin{pmatrix}0\\0\\*\end{pmatrix}^* \begin{pmatrix}0\\1\\*\end{pmatrix} \begin{pmatrix}0\\0\\*\end{pmatrix}^*$$

$$L_2 = \mathsf{drop}_1^{-1}(L_a(X(y))) = \left[\begin{pmatrix}*\\0\\0\end{pmatrix} + \begin{pmatrix}*\\0\\1\end{pmatrix}\right]^* \begin{pmatrix}*\\1\\1\end{pmatrix} \left[\begin{pmatrix}*\\0\\0\end{pmatrix} + \begin{pmatrix}*\\0\\1\end{pmatrix}\right]^*$$

We have $L_a(\psi) = (L_1 \cap L_2) \cap L(\mathcal{A}_{2,1})$. Here $\mathcal{A}_{2,1}$ is a DFA to check the admissibility condition. For $\chi = \exists y.\, y < x \wedge X(y)$ we obtain $L_a(\chi) = \mathsf{stz}(\mathsf{drop}_2(L_a(\psi)))$. Finally, $L_a(\varphi) = \mathsf{stz}(\mathsf{drop}_3(L_a(\chi)))$.

3 (a) We have $\varphi = \neg(\neg a \,\mathsf{U}\, \mathsf{X}\, b) \wedge \neg \mathsf{G}\, b \equiv (a \,\mathsf{R}\, \neg\mathsf{X}\, b) \wedge \mathsf{F}\, \neg b \equiv (a \,\mathsf{R}\, \mathsf{X}\, \neg b) \wedge (\top \,\mathsf{U}\, \neg b)$.

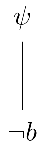
(b) Starting from the negation normal form $\psi$ of part (a) we obtain the ABA

$$A_\psi = (\mathcal{C}_+(\psi), 2^{\mathsf{AP}}, \Delta, \psi, \{a \,\mathsf{R}\, \mathsf{X}\, \neg b\})$$

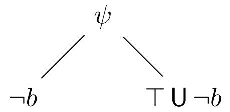
with $\mathcal{C}_+(\psi) = \{a, b, \neg b, \mathsf{X}\,\neg b, a \,\mathsf{R}\,\mathsf{X}\,\neg b, \top, \top\,\mathsf{U}\,\neg b, \psi\}$ and $\Delta$ defined as follows:

| | $\varnothing$ | $\{a\}$ | $\{b\}$ | $\{a, b\}$ |
|---|---|---|---|---|
| $a$ | $\bot$ | $\top$ | $\bot$ | $\top$ |
| $b$ | $\bot$ | $\bot$ | $\top$ | $\top$ |
| $\neg b$ | $\top$ | $\top$ | $\bot$ | $\bot$ |
| $\mathsf{X}\,\neg b$ | $\neg b$ | $\neg b$ | $\neg b$ | $\neg b$ |
| $a\,\mathsf{R}\,\mathsf{X}\,\neg b$ | $\neg b \wedge a\,\mathsf{R}\,\mathsf{X}\,\neg b$ | $\neg b$ | $\neg b \wedge a\,\mathsf{R}\,\mathsf{X}\,\neg b$ | $\neg b$ |
| $\top$ | $\top$ | $\top$ | $\top$ | $\top$ |
| $\top\,\mathsf{U}\,\neg b$ | $\top$ | $\top$ | $\top\,\mathsf{U}\,\neg b$ | $\top\,\mathsf{U}\,\neg b$ |
| $\psi$ | $\neg b \wedge a\,\mathsf{R}\,\mathsf{X}\,\neg b$ | $\neg b$ | $\neg b \wedge a\,\mathsf{R}\,\mathsf{X}\,\neg b \wedge \top\,\mathsf{U}\,\neg b$ | $\neg b \wedge \top\,\mathsf{U}\,\neg b$ |

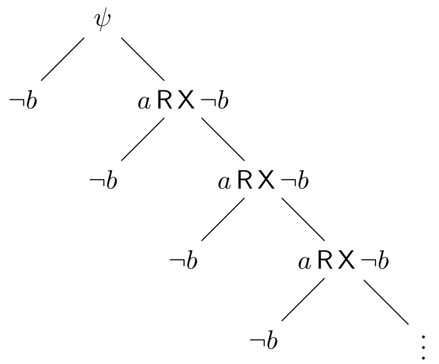
(c) The trace $\{a\}\{b\}^\omega$ is not accepted since




cannot be completed to an (accepting) run since the second input symbol is $\{b\}$. The trace $(\{a, b\}\varnothing\{a, b\})^\omega$ admits the accepting run




Finally, the trace $\varnothing^\omega$ admits the accepting run

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1. False.
2. False.
3. True.
4. False.
5. False.
6. False.
7. True.
8. True.
9. False.
10. True.