universität innsbruck

Automata and Logic	WS 2023/2024	LVA 703302

2nd EXAM – SOLUTIONS February 23, 2024

1 For instance, consider $A = \{a^n b^n \mid n \geqslant 0\}$ and $h(a) = h(b) = a$. We have $h(A) = \{a^{2n} \mid n \geqslant 0\}$ and $h^{-1}(A) = \{\epsilon\}$. Clearly, $h(A)$ and $h^{-1}(A)$ are regular but A is not.

2 Let $\Sigma = \{a\}$ and let $M_i = (Q_i, \Sigma, \delta_i, s_i, \{s_i\})$ be the DFA that counts modulo i, so

$$L(M_i) = \{x \in \{a\}^* \mid |x| = 0 \mod i\}$$

We use M_3, M_5 and M_7, and assume these three automata do not share states. The set

$$\{x \in \{a\}^* \mid |x| = 2 \mod 105\}$$

is accepted by the AFA $M = (Q, \Sigma, \Delta, s, F)$ with $Q = \{s, t\} \uplus Q_3 \uplus Q_5 \uplus Q_7$, $F = \{s_3, s_5, s_7\}$ and

$$\Delta(q, a) = \begin{cases} t & \text{if } q = s \\ s_3 \wedge s_5 \wedge s_7 & \text{if } q = t \\ \delta_i(q, a) & \text{if } q \in Q_i \text{ for } i \in \{3, 5, 7\} \end{cases}$$

3 For instance,

$$\psi = \exists \ell. \neg P_a(\ell) \wedge \neg P_b(\ell) \wedge \big(\forall x. \neg P_a(x) \wedge \neg P_b(x) \to \ell \leqslant x\big) \wedge \\ \exists X. X(0) \wedge \big(\forall x. \forall y. y = x + 1 \wedge y \leqslant \ell \to (X(x) \leftrightarrow \neg X(y))\big) \wedge X(\ell)$$

4 (a) The string 0011 does not belong to $L(\varphi)$ since $(1100)_2 = 12$ and $2x - 3y + 2z$ is even for all $x, z \in \mathbb{N}$. The string 1001 belongs to $L(\varphi)$ since $(1001)_2 = 9$ and $2x - 3y + 2z = 1$ for $x = 14$ and $z = 0$. The string 1000 belongs to $L(\varphi)$ since $(0001)_2 = 1$ and $2x - 3y + 2z = 1$ for $x = 2$ and $z = 0$.

(b) The construction from the lecture yields the DFA $M = (Q, \Sigma_3, \delta, s, F)$ with $Q = \{-3, -2, -1, 0, 1, 2, \bot\}$, $\Sigma = \{0, 1\}$, $s = 1$, $F = \{0\}$ and the following transition function δ:

	$\binom{0}{0}_0$	$\binom{1}{0}_0$	$\binom{0}{1}_0$	$\binom{0}{0}_1$	$\binom{1}{1}_0$	$\binom{1}{0}_1$	$\binom{0}{1}_1$	$\binom{1}{1}_1$
1	$\bot$	$\bot$	2	$\bot$	1	$\bot$	1	0
2	1	0	$\bot$	0	$\bot$	-1	$\bot$	$\bot$
0	0	-1	$\bot$	-1	$\bot$	-2	$\bot$	$\bot$
-1	$\bot$	$\bot$	1	$\bot$	0	$\bot$	0	-1
-2	-1	-2	$\bot$	-2	$\bot$	-3	$\bot$	$\bot$
-3	$\bot$	$\bot$	0	$\bot$	-1	$\bot$	-1	-2
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

(c) Applying the homomorphism Π_3 to the DFA M of part (b) yields the NFA $N = (Q, \Sigma_2, \Delta, S, F)$ with $S = \{1\}$ and Δ given in the following table:

	$\binom{0}{0}$	$\binom{1}{0}$	$\binom{0}{1}$	$\binom{1}{1}$
1	$\{\bot\}$	$\{\bot\}$	$\{1, 2\}$	$\{0, 1\}$
2	$\{0, 1\}$	$\{-1, 0\}$	$\{\bot\}$	$\{\bot\}$
0	$\{-1, 0\}$	$\{-2, -1\}$	$\{\bot\}$	$\{\bot\}$
-1	$\{\bot\}$	$\{\bot\}$	$\{0, 1\}$	$\{-1, 0\}$
-2	$\{-2, -1\}$	$\{-3, -2\}$	$\{\bot\}$	$\{\bot\}$
-3	$\{\bot\}$	$\{\bot\}$	$\{-1, 0\}$	$\{-2, -1\}$
$\bot$	$\{\bot\}$	$\{\bot\}$	$\{\bot\}$	$\{\bot\}$

Applying the homomorphism Π_1 to the NFA N yields the NFA $N' = (Q, \Sigma, \Delta', S, F)$ with Δ' given in the following table:

	0	1
1	$\{\bot\}$	$\{0,1,2\}$
2	$\{-1,0,1\}$	$\{\bot\}$
0	$\{-2,-1,0\}$	$\{\bot\}$
-1	$\{\bot\}$	$\{-1,0,1\}$
-2	$\{-3,-2,-1\}$	$\{\bot\}$
-3	$\{\bot\}$	$\{-2,-1,0\}$
$\bot$	$\{\bot\}$	$\{\bot\}$

This latter NFA accepts all strings starting with a 1. These strings represent all odd numbers and every odd number satisfies φ.

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1. False.
2. True.
3. True.
4. True.
5. False.
6. True.
7. True.
8. False.
9. False.
10. False.