1 (a) A set $A \subseteq \mathbb{N}$ is diophantine if there exists a polynomial $P(x, y_1, \dots, y_n)$ with integer coefficients such that

$$x \in A \iff \exists\, y_1 \ \cdots\ \exists\, y_n \ P(x, y_1, \dots, y_n) = 0$$

(b) We have $\{x^2 \mid x \text{ is odd}\} = \{x \mid x - (2y+1)^2 = 0\}$ and thus the polynomial $P(x, y) = x - 4y^2 - 4y - 1$ shows that the set $\{x^2 \mid x \text{ is odd}\}$ is diophantine.

(c) Let $A = \{x \mid P(x, y_1, \dots, y_n) = 0 \text{ for some } y_1, \dots, y_n \in \mathbb{N}\}$ be an arbitrary diophantine set and consider the partial recursive function

$$\varphi(x) = (\mu\, y)\,(P(x, (y)_1, \dots, (y)_n)^2 = 0)$$

Since A is the domain of φ, A is recursively enumerable.

2 (a) Given a combinator A for addition, we can take $\mathsf{double} = \langle x\rangle(\mathsf{A}\,x\,x) = \mathsf{S\,A\,I}$ because

$$\mathsf{S\,A\,I}\,\underline{n} \to \mathsf{A}\,\underline{n}\,(\mathsf{I}\,\underline{n}) \to \mathsf{A}\,\underline{n}\,(\mathsf{I}\,\underline{n}) \to \mathsf{A}\,\underline{n}\,\underline{n} \to^* \underline{n+n} = \underline{2n}$$

There are many combinators that represent addition, e.g. $\mathsf{A} = \mathsf{C\,I\,(S\,B)}$.

(b) Let us write ω for SII. We have

$$\begin{aligned}
&\mathsf{S(K}\omega\mathsf{)(S(S(KS)K)(K}\omega\mathsf{))}x \\
&\quad\to \mathsf{K}\omega x\mathsf{(S(S(KS)K)(K}\omega\mathsf{)}x) \\
&\quad\to \omega\mathsf{(S(S(KS)K)(K}\omega\mathsf{)}x) \\
&\quad\to^+ \mathsf{S(S(KS)K)(K}\omega\mathsf{)}x\mathsf{(S(S(KS)K)(K}\omega\mathsf{)}x) \\
&\quad\to \mathsf{S(KS)K}x\mathsf{(K}\omega x\mathsf{)(S(S(KS)K)(K}\omega\mathsf{)}x) \\
&\quad\to \mathsf{KS}x\mathsf{(K}x\mathsf{)(K}\omega x\mathsf{)((S(S(KS)K)(K}\omega\mathsf{))}x) \\
&\quad\to \mathsf{S(K}x\mathsf{)(K}\omega x\mathsf{)((S(S(KS)K)(K}\omega\mathsf{))}x) \\
&\quad\to \mathsf{K}x\mathsf{(S(S(KS)K)(K}\omega\mathsf{)}x\mathsf{)(K}\omega x\mathsf{)(S(S(KS)K)(K}\omega\mathsf{)}x) \\
&\quad\to x\mathsf{(K}\omega x\mathsf{)(S(S(KS)K)(K}\omega\mathsf{)}x) \\
&\quad\leftarrow x\mathsf{(S(K}\omega\mathsf{)(S(S(KS)K)(K}\omega\mathsf{))}x)
\end{aligned}$$

(c) Decomposing $\mathsf{SI(KI)}$ gives the types

$\mathsf{SI(KI)}\colon \beta$	$\mathsf{SI}\colon \alpha \to \beta$	$\mathsf{S}\colon \gamma \to \alpha \to \beta$	$\mathsf{I}\colon \gamma$
	$\mathsf{KI}\colon \alpha$	$\mathsf{K}\colon \delta \to \alpha$	$\mathsf{I}\colon \delta$

with the following constraints

$$\gamma \to \alpha \to \beta \approx (\rho_1 \to \sigma_1 \to \tau_1) \to (\rho_1 \to \sigma_1) \to \rho_1 \to \tau_1 \qquad \gamma \approx \sigma_2 \to \sigma_2$$
$$\delta \to \alpha \approx \sigma_3 \to \tau_3 \to \sigma_3 \qquad \delta \approx \sigma_4 \to \sigma_4$$

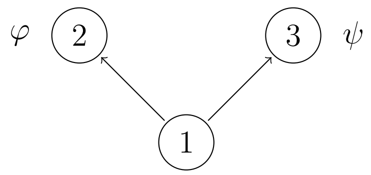
Solving these constraints with the unification algorithm produces the mgu with $\beta \mapsto ((\sigma_4 \to \sigma_4) \to \tau_1) \to \tau_1$.

3 (a) We have $\{\varphi \to \psi, \psi \to \chi, \varphi\} \vdash_{\mathsf{h}} \chi$:

1.	$\varphi \to \psi$	assumption
2.	φ	assumption
3.	ψ	modus ponens 1, 2
4.	$\psi \to \chi$	assumption
5.	χ	modus ponens 4, 3

Hence $\vdash_{\mathsf{h}} (\varphi \to \psi) \to (\psi \to \chi) \to \varphi \to \chi$ by three applications of the deduction theorem. It follows that $(\varphi \to \psi) \to (\psi \to \chi) \to \varphi \to \chi$ is intuitionistically valid.

(b) Consider the Kripke model $\mathcal{C}$



From the table

	φ	ψ	$\varphi \wedge \psi$	$\neg(\varphi \wedge \psi)$	$\neg\varphi$	$\neg\psi$	$\neg\varphi \vee \neg\psi$	$\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$
1	×	×	×	✓	×	×	×	×
2	✓	×	×	✓	×	✓	✓	✓
3	×	✓	×	✓	✓	×	✓	✓

we infer $\mathcal{C}, 1 \not\Vdash \neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$ and thus $\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$ is not intuitionistically valid.

(c) First of all, $\{(\varphi \vee \psi) \wedge \neg\psi, \psi\} \vdash_{\mathsf{h}} \varphi$:

1.	$(\varphi \vee \psi) \wedge \neg\psi$	assumption
2.	$(\varphi \vee \psi) \wedge \neg\psi \to (\varphi \vee \psi)$	axiom 3
3.	$\varphi \vee \psi$	modus ponens 2, 1
4.	$(\varphi \vee \psi) \wedge \neg\psi \to (\psi \to \bot)$	axiom 4
5.	$\neg\psi$	modus ponens 4, 1
6.	ψ	assumption
7.	$\bot$	modus ponens 5, 6
8.	$\bot \to \varphi$	axiom 9
9.	φ	modus ponens 8, 7

Hence $(\varphi \vee \psi) \wedge \neg\psi \vdash_{\mathsf{h}} \psi \to \varphi$ $(\star)$ by the deduction theorem. Next we show $(\varphi \vee \psi) \wedge \neg\psi \vdash_{\mathsf{h}} \varphi$:

1.	$(\varphi \vee \psi) \wedge \neg\psi$	assumption
2.	$\varphi \vee \psi$	*line 3 above*
3.	$\psi \to \varphi$	$(\star)$
4.	$\varphi \to \varphi$	*theorem*
5.	$(\varphi \to \varphi) \to (\psi \to \varphi) \to \varphi \vee \psi \to \varphi$	axiom 8
6.	$(\psi \to \varphi) \to \varphi \vee \psi \to \varphi$	modus ponens 5, 4
7.	$\varphi \vee \psi \to \varphi$	modus ponens 6, 3
8.	φ	modus ponens 7, 2

Hence $\vdash_{\mathsf{h}} (\varphi \vee \psi) \wedge \neg\psi \to \varphi$ by the deduction theorem. It follows that $(\varphi \vee \psi) \wedge \neg\psi \to \varphi$ is intuitionistically valid.

4 (a) Let A be a non-trivial index set. So there exist numbers $d \in A$ and $e \notin A$. For a proof by contradiction, suppose A is recursive. Hence the function

$$f(x) = \begin{cases} e & \text{if } x \in A \\ d & \text{if } x \notin A \end{cases}$$

is recursive. The fixed point theorem yields a number a such that $\varphi_a \simeq \varphi_{fa}$. We distinguish two cases.

i. If $a \in A$ then $f(a) \in A$ because A is an index set but $f(a) = e \notin A$.

ii. If $a \notin A$ then $f(a) \notin A$ because A is an index set but $f(a) = d \in A$.

In both cases we have a contradiction. Hence A is not recursive.

(b) We distinguish five cases for $t \to u$.

i. Suppose $t = \mathsf{I}\, t_1 \to t_1 = u$. From $\Gamma \vdash t : \tau$ we infer $\vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma$. Hence $\sigma = \tau$ and thus $\Gamma \vdash u : \tau$.

ii. Suppose $t = \mathsf{K}\, t_1 t_2 \to t_1 = u$. From $\Gamma \vdash t : \tau$ we infer $\Gamma \vdash \mathsf{K}\, t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$. The former entails $\vdash \mathsf{K} : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \rho$. Hence $\rho = \tau$ and thus $\Gamma \vdash u : \tau$.

iii. Suppose $t = \mathsf{S}\, t_1 t_2 t_3 \to t_1 t_3 (t_2 t_3) = u$. From $\Gamma \vdash t : \tau$ we infer $\Gamma \vdash \mathsf{S}\, t_1 t_2 : \sigma \to \tau$ and $\Gamma \vdash t_3 : \sigma$. The former entails $\Gamma \vdash \mathsf{S}\, t_1 : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_2 : \rho$. Further, $\vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \mu$. From $\vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ we obtain $\rho = \sigma \to \rho_1$ and $\mu = \sigma \to \rho_1 \to \tau$. Hence $\Gamma \vdash t_1 t_3 : \rho_1 \to \tau$ and $\Gamma \vdash t_2 t_3 : \rho_1$ and therefore $\Gamma \vdash u : \tau$.

iv. Suppose $t = t_1 t_2 \to u_1 t_2 = u$ with $t_1 \to u_1$. From $\Gamma \vdash t : \tau$ we infer $\Gamma \vdash t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$. We obtain $\Gamma \vdash u_1 : \sigma \to \tau$ from the induction hypothesis. Hence $\Gamma \vdash u : \tau$.

v. Suppose $t = t_1 t_2 \to t_1 u_2 = u$ with $t_2 \to u_2$. From $\Gamma \vdash t : \tau$ we infer $\Gamma \vdash t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$. We obtain $\Gamma \vdash u_2 : \sigma$ from the induction hypothesis. Hence $\Gamma \vdash u : \tau$.

(c) We use induction on x. If $x = 0$ then $y = 1$. Clearly, $x = F_0$ and $y = F_1$ are consecutive Fibonacci numbers. Suppose $x > 0$. We have $yx + x^2 > y$ and thus $1 = y^2 - (yx + x^2) < y^2 - y$. Hence $y \geqslant 2$. Consequently,

$$(x+1)^2 = x^2 + 2x + 1 \leqslant x^2 + yx + 1 = y^2$$

and thus $y > x$. Hence

$$y^2 = yx + x^2 + 1 \leqslant yx + x^2 + x = yx + (x+1)x \leqslant yx + yx = 2yx$$

and therefore $y \leqslant 2x$. Now let $a = 2x - y$ and $b = y - x$. We have $0 \leqslant a < x$ and $0 < b$. Moreover

$$b^2 - ba - a^2 = (y-x)^2 - (y-x)(2x-y) - (2x-y)^2 = y^2 - yx - x^2 = 1$$

Since $a < x$ we can apply the induction hypothesis. This yields $a = F_i$ and $b = F_{i+1}$ for some $i \geqslant 0$. Hence $x = a + b = F_{i+2}$ and $y = b + x = F_{i+3}$.

5 The second and third statements are true, the others are false.