This test consists of five exercises. The available points for each item are written in the margin. ***Explain your answers to the first four exercises!***

⟨6⟩ **1** (a) Recall the definition of a diophantine set of natural numbers.

⟨9⟩ (b) Prove that the set $\{x^2 \mid x \text{ is odd}\}$ is diophantine.

⟨9⟩ (c) Prove that diophantine sets are recursively enumerable.

2 This exercise is about combinatory logic.

⟨8⟩ (a) Construct a combinator double such that $\mathsf{double}\ \underline{n} \to^* \underline{2n}$ for all $n \geqslant 0$.

⟨8⟩ (b) Prove that $\mathsf{S}(\mathsf{K}(\mathsf{SII}))(\mathsf{S}(\mathsf{S}(\mathsf{KS})\mathsf{K})(\mathsf{K}(\mathsf{SII})))$ is a fixed point combinator.

⟨8⟩ (c) Compute the principle type of $\mathsf{SI}(\mathsf{KI})$.

3 Which of the following propositional formulas are intuitionistically valid? For those that are, provide a Hilbert-style proof. For those that are not, construct a Kripke model that shows this.

⟨9⟩ (a) $(\varphi \to \psi) \to (\psi \to \chi) \to \varphi \to \chi$

⟨9⟩ (b) $\neg(\varphi \land \psi) \to (\neg\varphi \lor \neg\psi)$

⟨9⟩ (c) $((\varphi \lor \psi) \land \neg\psi) \to \varphi$

⟨15⟩ **4** Prove ***one of*** the following statements.

(a) Non-trivial index sets are not recursive.

(b) In typed combinatory logic, if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$.

(c) If $y^2 - yx - x^2 = 1$ for $x, y \geqslant 0$ then x and y are consecutive Fibonacci numbers.

⟨10⟩ **5** Determine whether the following statements are true or false. Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

1. $\lambda x.\, y\, (\lambda y.\, x\, y\, z) \equiv_\alpha \lambda y.\, x\, (\lambda x.\, y\, x\, z)$
2. The function $\mathsf{fib}(\mathsf{fib}(x))$ belongs to E_4.
3. Every typable CL-term is strongly computable.
4. The unary function $\varphi_{\langle 5,\langle 3,\langle 1\rangle,\langle 5,\langle 2,1,1\rangle\rangle\rangle\rangle}$ is LOOP computable.
5. The partial recursive function $\psi(x) = \varphi_x(x) + 1$ can be extended to a total recursive function.