Exercises

⟨1⟩ 1. Prepare for the test.

⟨1⟩ 2. A fixed point combinator is a closed lambda term Y such that $Y\,F$ is a fixed point of F, for every lambda term F. Consider the following lambda terms:

$$\begin{aligned} ¢ &\equiv \lambda abcdefghijklmnopqstuvwxyzr.r(thisisafixedpointcombinator) \\ \$ &\equiv ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢ \end{aligned}$$

Prove that \$ is a fixed point combinator.

3. In this exercise we consider an alternative to Church numerals. Define

$$\mathsf{I} \equiv \lambda x.\,x \quad \mathsf{T} \equiv \lambda xy.\,x \quad \mathsf{F} \equiv \lambda xy.\,y \quad [M,N] \equiv \lambda z.\,z\,M\,N \quad \ulcorner n \urcorner \equiv \begin{cases} \mathsf{I} & \text{if } n = 0 \\ [\mathsf{F}, \ulcorner n-1 \urcorner] & \text{if } n > 0 \end{cases}$$

for all lambda terms M, N and $n \in \mathbb{N}$.

⟨2⟩ (a) Construct combinators S, P and Z such that

$$\mathsf{S}\,\ulcorner n \urcorner =_\beta \ulcorner n+1 \urcorner \qquad \mathsf{P}\,\ulcorner n \urcorner =_\beta \begin{cases} \ulcorner 0 \urcorner & \text{if } n = 0 \\ \ulcorner n-1 \urcorner & \text{if } n > 0 \end{cases} \qquad \mathsf{Z}\,\ulcorner n \urcorner =_\beta \begin{cases} \mathsf{T} & \text{if } n = 0 \\ \mathsf{F} & \text{if } n > 0 \end{cases}$$

for all $n \geqslant 0$.

⟨2⟩ (b) Construct combinators F and G such that

$$F\,\underline{n} =_\beta \ulcorner n \urcorner \qquad\qquad G\,\ulcorner n \urcorner =_\beta \underline{n}$$

for all $n \geqslant 0$. Here $\underline{n}$ denotes the n-th Church numeral.

⟨1⟩ 4. Given lambda terms F_1 and F_2, construct lambda terms X_1 and X_2 such that

$$X_1 \to_\beta^* F_1\,X_1\,X_2 \qquad\qquad X_2 \to_\beta^* F_2\,X_1\,X_2$$