WS 2023 lecture 2

Computability Theory

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Outline

Definition

class PR of primitive recursive functions is smallest class of total functions $f\colon \mathbb{N}^n \to \mathbb{N}$ that contains all initial functions

- zero $\mathrm{z}(x) = 0$
- successor $\mathrm{s}(x) = x + 1$
- projection $\pi_i^n(x_1, \dots, x_n) = x_i$ for all $n \geqslant 1$ and $1 \leqslant i \leqslant n$

and is closed under composition

- $f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x})) \in \mathrm{PR}$ for all $g\colon \mathbb{N}^m \to \mathbb{N} \in \mathrm{PR}$ and $h_1, \dots, h_m\colon \mathbb{N}^n \to \mathbb{N} \in \mathrm{PR}$

and primitive recursion

- $f(x, \vec{y})\colon \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}) \\ f(x+1, \vec{y}) &= h(f(x, \vec{y}), x, \vec{y}) \end{aligned}$$

belongs to PR for all $g\colon \mathbb{N}^n \to \mathbb{N} \in \mathrm{PR}$ and $h\colon \mathbb{N}^{n+2} \to \mathbb{N} \in \mathrm{PR}$

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is primitive recursive if its characteristic function $\chi_P\colon \mathbb{N}^n \to \mathbb{N}$

$$\chi_P(x_1,\dots,x_n) = \begin{cases} 1 & \text{if } P(x_1,\dots,x_n) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive

Lemma

PR is closed under iteration, case analysis and bounded minimization

Definitions

- pairing function $\pi(x,y) = 2^x(2y+1) \dot{-} 1$
- extraction functions

$$\pi_1(z) = (\mu\, x \leqslant z)\,(\exists\, y \leqslant z)\,\big[\, z = \pi(x,y) \,\big] \qquad \pi_2(z) = (\mu\, y \leqslant z)\,(\exists\, x \leqslant z)\,\big[\, z = \pi(x,y) \,\big]$$

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course−of−values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel’s β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s−m−n theorem, total recursive functions, undecidability, while programs, . . .

Part II: Combinatory Logic and Lambda Calculus

α−equivalence, abstraction, arithmetization, β−reduction, CL−representability, combinators, combinatorial completeness, Church numerals, Church−Rosser theorem, Curry−Howard isomorphism, de Bruijn notation, η−reduction, fixed point theorem, intuitionistic propositional logic, λ−definability, normalization theorem, termination, typing, undecidability, Z property, . . .

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1. **Summary of Previous Lecture**
2. **Gödel Numbering**
3. **Course–of–Values Recursion**
4. **Ackermann Function**
5. **Diagonalization**
6. **Total Recursive Functions**
7. **Summary**

Definitions

- Gödel number of sequence $x_1, \ldots, x_n$: $\langle x_1, \ldots, x_n \rangle = \mathsf{p}_0^n \times \mathsf{p}_1^{x_1} \times \cdots \times \mathsf{p}_n^{x_n}$

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Lemma

- $x = \langle x_1, \dots, x_n \rangle \implies \mathrm{len}(x) = n \wedge (x)_i = x_i$ for all $1 \leqslant i \leqslant n$

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- $x = \langle x_1, \ldots, x_n \rangle \wedge y = \langle y_1, \ldots, y_m \rangle \implies x \,;\, y = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$

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if $g\colon \mathbb{N}^{n} \to \mathbb{N}$ and $h\colon \mathbb{N}^{n+2} \to \mathbb{N}$ are primitive recursive then so is $f\colon \mathbb{N}^{n+1} \to \mathbb{N}$ defined as

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with $h'(x,y,\vec{z}) = x \mathbin{;} \langle h(x,y,\vec{z})\rangle$

Outline

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$$\begin{aligned} \mathrm{ack}(0,y) &= y+1 \\ \mathrm{ack}(x+1,0) &= \mathrm{ack}(x,1) \\ \mathrm{ack}(x+1,y+1) &= \mathrm{ack}(x,\mathrm{ack}(x+1,y)) \end{aligned}$$

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Example

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\end{aligned}
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Example

- $\mathsf{ack}(1, 3) = \mathsf{ack}(0, 4)$

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- $\mathsf{ack}(1,3) = 5$

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Example

- $\mathrm{ack}(1,3) = 5$
- $\mathrm{ack}(2,2)$

Definition

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- $\mathrm{ack}(2, 2) = \mathrm{ack}(1, \mathrm{ack}(2, 1))$

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Example

- $\mathrm{ack}(1,3) = 5$
- $\mathrm{ack}(2,2) = \mathrm{ack}(1,5)$

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Example

- $\mathrm{ack}(1, 3) = 5$
- $\mathrm{ack}(2, 2) = \mathrm{ack}(0, \mathrm{ack}(1, 4))$

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- $\mathrm{ack}(1,3) = 5$
- $\mathrm{ack}(2,2) = \mathrm{ack}(0,6)$

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- $\mathrm{ack}(1,3) = 5$
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Example

- $\mathrm{ack}(1,3) = 5$
- $\mathrm{ack}(2,2) = 7$
- $\mathrm{ack}(3,4) = 125$

Definition

$$\begin{aligned} \mathrm{ack}(0,y) &= y+1 \\ \mathrm{ack}(x+1,0) &= \mathrm{ack}(x,1) \\ \mathrm{ack}(x+1,y+1) &= \mathrm{ack}(x,\mathrm{ack}(x+1,y)) \end{aligned}$$

Example

- $\mathrm{ack}(1,3) = 5$
- $\mathrm{ack}(2,2) = 7$
- $\mathrm{ack}(3,4) = 125$
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Lemma

Ackermann function is well-defined

Example (cont'd)

- $\mathrm{ack}(4,3) = 2^{2^{65533}} - 3$

Example (cont'd)

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Remarks

- Ackermann function grows very fast

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Example (cont'd)

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Remarks

- Ackermann function grows very fast
- Ackermann function is used as compiler benchmark
- inverse of Ackermann function appears in complexity analysis of certain algorithms

Lemma

① $\mathsf{ack}(x,y) > y$

② $\mathsf{ack}(x,y+1) > \mathsf{ack}(x,y)$

③ $\mathsf{ack}(x+1,y) \geqslant \mathsf{ack}(x,y+1)$

④ $\mathsf{ack}(x+2,y) > \mathsf{ack}(x,2y)$

▸ skip proofs

Lemma ①

- $\mathsf{ack}(x, y) > y$

Proof

induction on x

- $\mathsf{ack}(0, y) = y + 1 > y$

Lemma ①

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Proof

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- $\mathsf{ack}(0, y) = y + 1 > y$
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$$\mathsf{ack}(x + 1, 0) = \mathsf{ack}(x, 1)$$

Lemma ①

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Proof

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- $\mathrm{ack}(0, y) = y + 1 > y$
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$$\mathrm{ack}(x + 1, 0) = \mathrm{ack}(x, 1) > 1$$

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$$\mathrm{ack}(x+1, 0) = \mathrm{ack}(x, 1) > 1 > 0$$
$$\mathrm{ack}(x+1, y+1) = \mathrm{ack}(x, \mathrm{ack}(x+1, y))$$

Lemma ①

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- $\mathsf{ack}(0, y) = y + 1 > y$
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Lemma ❷

- $\mathsf{ack}(x, y+1) > \mathsf{ack}(x, y)$

Proof

induction on x

$$\mathsf{ack}(0, y+1) = y+2 > y+1 = \mathsf{ack}(0, y)$$

Lemma ②

- $\mathsf{ack}(x, y+1) > \mathsf{ack}(x, y)$

Proof

induction on x

$$\mathsf{ack}(0, y+1) = y+2 > y+1 = \mathsf{ack}(0, y)$$

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Proof

induction on x

Lemma ③

- $\mathsf{ack}(x+1,y) \geqslant \mathsf{ack}(x,y+1)$

Proof

induction on x

- $\mathsf{ack}(1,y) \geqslant y+2 = \mathsf{ack}(0,y+1)$ by induction on y

Lemma ③

- $\mathsf{ack}(x+1,y) \geqslant \mathsf{ack}(x,y+1)$

Proof

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$$\mathsf{ack}(1,0) = \mathsf{ack}(0,1) = 2$$

Lemma ③

- $\text{ack}(x+1,y) \geqslant \text{ack}(x,y+1)$

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- induction on y

$$\mathrm{ack}(x+2,0) = \mathrm{ack}(x+1,1)$$

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- $\mathsf{ack}(x+1, y) \geqslant \mathsf{ack}(x, y+1)$

Proof

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Lemma ③

- $\mathrm{ack}(x+1,y) \geqslant \mathrm{ack}(x,y+1) > \mathrm{ack}(x,y)$

Corollary

Ackermann function is strictly monotone in both arguments

Lemma ④

- $\mathsf{ack}(x+2, y) > \mathsf{ack}(x, 2y)$

Proof

induction on x

Lemma ④

- $\mathsf{ack}(x+2, y) > \mathsf{ack}(x, 2y)$

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- $\mathsf{ack}(2, y) > \mathsf{ack}(0, 2y)$ by induction on y

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- $\mathrm{ack}(2,y) > \mathrm{ack}(0,2y)$ by induction on y

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$$\begin{aligned} \mathrm{ack}(2,0) &\geqslant \mathrm{ack}(1,1) \geqslant \mathrm{ack}(0,2) > \mathrm{ack}(0,1) > \mathrm{ack}(0,0) \\ \mathrm{ack}(2,y+1) &= \mathrm{ack}(1,\mathrm{ack}(2,y)) \end{aligned}$$

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$$\mathrm{ack}(2,y) > \mathrm{ack}(0,2y) > 2y$$

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$\mathsf{ack}(2,y) > 2y+1$

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- induction on y

$$\mathrm{ack}(x+3,0) \geqslant \mathrm{ack}(x+2,1) \geqslant \mathrm{ack}(x+1,2)$$

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Lemma ④

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Lemma

$\forall$ primitive recursive function $f : \mathbb{N}^n \to \mathbb{N}$ $\exists$ constant $c \in \mathbb{N}$ such that

$$f(x_1, \ldots, x_n) < \mathrm{ack}(c, \max\{x_1, \ldots, x_n\})$$

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induction on definition of primitive recursive functions

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induction on definition of primitive recursive functions

- initial functions

  $\mathrm{z}(x) = 0 < x + 1 = \mathrm{ack}(0, x)$

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  $$\mathsf{s}(x) = x + 1 = \mathsf{ack}(0, x) < \mathsf{ack}(1, x)$$

Lemma

$\forall$ primitive recursive function $f\colon \mathbb{N}^n \to \mathbb{N}$ $\exists$ constant $c \in \mathbb{N}$ such that

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Proof

induction on definition of primitive recursive functions

- initial functions

  $\mathsf{z}(x) = 0 < x + 1 = \mathsf{ack}(0,x)$

  $\mathsf{s}(x) = x + 1 = \mathsf{ack}(0,x) < \mathsf{ack}(1,x)$

  $\pi^n_i(x_1,\dots,x_n) = x_i \leqslant \max\{x_1,\dots,x_n\} < \mathsf{ack}(0, \max\{x_1,\dots,x_n\})$

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- composition $f(\vec{x}) = g(h_1(\vec{x}),\dots,h_m(\vec{x}))$

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Theorem

Ackermann function is not primitive recursive

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if Ackermann function is primitive recursive then

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Outline

1. **Summary of Previous Lecture**
2. **Gödel Numbering**
3. **Course–of–Values Recursion**
4. **Ackermann Function**
5. **Diagonalization**
6. **Total Recursive Functions**
7. **Summary**

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Remark (key insight)

from index we can reconstruct function

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Corollary

$\exists$ computable list $f_0, f_1, f_2, f_3, \ldots$ of unary primitive recursive functions

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Diagonalization

$$\begin{array}{ccccc}
f_0(0) & f_0(1) & f_0(2) & f_0(3) & \cdots \\
f_1(0) & f_1(1) & f_1(2) & f_1(3) & \cdots \\
f_2(0) & f_2(1) & f_2(2) & f_2(3) & \cdots \\
f_3(0) & f_3(1) & f_3(2) & f_3(3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Corollary

$\exists$ computable list $f_0, f_1, f_2, f_3, \ldots$ of unary primitive recursive functions

Diagonalization

$$\begin{array}{ccccc}
f_0(0) & f_0(1) & f_0(2) & f_0(3) & \cdots \\
f_1(0) & f_1(1) & f_1(2) & f_1(3) & \cdots \\
f_2(0) & f_2(1) & f_2(2) & f_2(3) & \cdots \\
f_3(0) & f_3(1) & f_3(2) & f_3(3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$$g(x) = f_x(x) + 1$$

Corollary

$\exists$ computable list $f_0, f_1, f_2, f_3, \ldots$ of unary primitive recursive functions

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$$g(x) = f_x(x) + 1$$

Corollary

function g is computable but not primitive recursive

Outline

1. Summary of Previous Lecture
2. Gödel Numbering
3. Course–of–Values Recursion
4. Ackermann Function
5. Diagonalization
6. **Total Recursive Functions**
7. Summary

Definition

class R of recursive functions is smallest class of total functions that contains all initial functions and is closed under composition, primitive recursion, and (unbounded) minimization:

$$(\mu\, i)\, P(i, \vec{y}) = \min\{ i \mid P(i, \vec{y}) \} \in \mathrm{R}$$

for all $P \colon \mathbb{N}^{n+1} \to \mathbb{B}$ such that $\chi_P \in \mathrm{R}$

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Theorem

Ackermann function is recursive

Representing Computations

$$\begin{aligned} \mathsf{ack}(0,y) &= y+1 \\ \mathsf{ack}(x+1,0) &= \mathsf{ack}(x,1) \\ \mathsf{ack}(x+1,y+1) &= \mathsf{ack}(x,\mathsf{ack}(x+1,y)) \end{aligned}$$

$\mathsf{ack}(1,2) = 4$

Representing Computations

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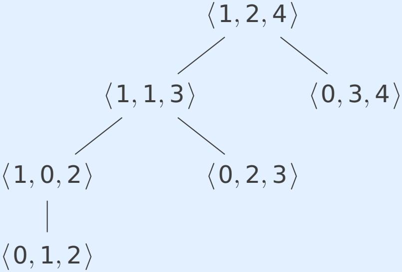
\end{aligned}
$$

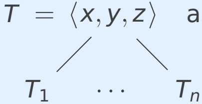
$\mathsf{ack}(1,2) = 4$:



$\langle x,y,z\rangle$ denotes $\mathsf{ack}(x,y) = z$

Definition

encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \ldots, \ulcorner T_n \urcorner \rangle$



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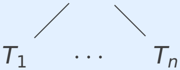

Lemma

$\exists$ primitive recursive predicate $P \colon \mathbb{N} \to \mathbb{B}$ such that

$$P(x) \iff x \text{ encodes correct computation tree}$$

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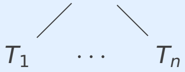


$T_1 \quad \cdots \quad T_n$

Lemma

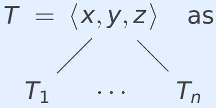
$\exists$ primitive recursive predicate $P \colon \mathbb{N} \to \mathbb{B}$ such that

$$P(x) \iff x \text{ encodes correct computation tree}$$

Corollary

- $\mathrm{ack}(x, y) = z \iff \exists\, t$ such that $P(t)$ and $(t)_1 = \langle x, y, z \rangle$

Definition



encode computation tree $T = \langle x, y, z \rangle$ with subtrees $T_1 \; \cdots \; T_n$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$

Lemma

$\exists$ primitive recursive predicate $P \colon \mathbb{N} \to \mathbb{B}$ such that

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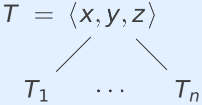
Corollary

- $\mathsf{ack}(x, y) = z \iff \exists\, t$ such that $P(t)$ and $(t)_1 = \langle x, y, z \rangle$
- $\mathsf{ack}(x, y) = \big((\mu\, t)\; (P(t) \wedge (t)_{1,1} = x \wedge (t)_{1,2} = y)\big)_{1,3}$

Proof

- $P(x) \iff \mathrm{seq}(x) \land \mathrm{seq}((x)_1) \land \mathrm{len}((x)_1) = 3 \land (A(x) \lor B(x) \lor C(x))$

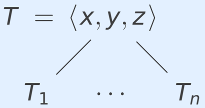


$$\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$$

Proof

- $P(x) \iff \mathrm{seq}(x) \wedge \mathrm{seq}((x)_1) \wedge \mathrm{len}((x)_1) = 3 \wedge (A(x) \vee B(x) \vee C(x))$
- $A(x) \iff \mathrm{len}(x) = 1 \wedge (x)_{1,1} = 0 \wedge (x)_{1,3} = \mathrm{s}((x)_{1,2})$

$$\mathrm{ack}(0, y) = y + 1$$



$T = \langle x, y, z \rangle$

$T_1 \quad \cdots \quad T_n$

$$\ulcorner T \urcorner = \langle \langle 0, y, y + 1 \rangle \rangle$$

Proof

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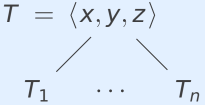
- $A(x) \iff \mathrm{len}(x) = 1 \wedge (x)_{1,1} = 0 \wedge (x)_{1,3} = \mathrm{s}((x)_{1,2})$
- $B(x) \iff \mathrm{len}(x) = 2 \wedge (x)_{1,1} > 0 \wedge (x)_{1,2} = 0 \wedge (x)_{2,1,1} = \mathrm{p}((x)_{1,1}) \wedge (x)_{2,1,2} = 1$
  $\wedge\ (x)_{2,1,3} = (x)_{1,3} \wedge P((x)_2)$

$$\mathrm{ack}(x+1,0) = \mathrm{ack}(x,1)$$



$$\ulcorner T \urcorner = \langle \langle x+1, 0, z \rangle, \langle \langle x, 1, z \rangle, \ldots \rangle \rangle$$

Proof

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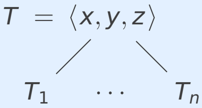
- $C(x) \iff \mathsf{len}(x) = 3 \wedge (x)_{1,1} > 0 \wedge (x)_{1,2} > 0 \wedge (x)_{2,1,1} = (x)_{1,1} \wedge (x)_{2,1,2} = \mathsf{p}((x)_{1,2})$
  $\wedge\ (x)_{3,1,1} = \mathsf{p}((x)_{1,1}) \wedge (x)_{3,1,2} = (x)_{2,1,3} \wedge (x)_{3,1,3} = (x)_{1,3} \wedge P((x)_2) \wedge P((x)_3)$

$$\mathsf{ack}(x+1, y+1) = \mathsf{ack}(x, \mathsf{ack}(x+1, y))$$



$T = \langle x, y, z \rangle$

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$$\ulcorner T \urcorner = \langle \langle x+1, y+1, z \rangle, \langle \langle x+1, y, w \rangle, \ldots \rangle, \langle \langle x, w, z \rangle, \ldots \rangle \rangle$$

Outline

1. **Summary of Previous Lecture**
2. **Gödel Numbering**
3. **Course–of–Values Recursion**
4. **Ackermann Function**
5. **Diagonalization**
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7. **Summary**

Important Concepts

- Ackermann function
- course−of−values recursion
- diagonalization
- Gödel numbering
- index
- R
- recursive function
- (unbounded) minimization

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- Ackermann function
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homework for October 16