universität innsbruck

WS 2023 lecture 5

Computability Theory

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Outline

1. **Summary of Previous Lecture**
2. **Normal Form Theorem**
3. **Undecidability**
4. **s–m–n or Parameterization Theorem**
5. **Fixed Point Theorem**
6. **Summary**

Theorem

class R of recursive functions is smallest class of total functions that contains all projection functions, addition and multiplication, characteristic function $\chi_=$ of equality predicate, and is closed under composition and minimization

Definition

$\beta(a,i) = \pi_1(a) \bmod \big(1 + (i+1)\,\pi_2(a)\big)$ Gödel's β function

Definition

class PA of partial recursive functions is smallest class of partial functions that contains all initial functions and is closed under composition, primitive recursion and unbounded minimization:

$$(\mu\, i)\,(f(i,\vec{y}) = 0) = \min\{\, i \mid f(i,\vec{y}) = 0 \text{ and } f(j,\vec{y}) > 0 \text{ for all } j < i \,\}$$

belongs to PA whenever $f\colon \mathbb{N}^{n+1} \to \mathbb{N}$ belongs to PA

Definition

function $f\colon \mathbb{N}^n \to \mathbb{N}$ is WHILE computable if $\exists$ WHILE program $P(x_1, \dots, x_n; y)$ such that $y = f(x_1, \dots, x_n)$ after execution of P

Theorem

function φ is partial recursive $\iff$ φ is WHILE computable

Definition

index $\ulcorner f \urcorner \in \mathbb{N}$ of derivation of partial recursive function f is defined inductively:

- $\ulcorner \mathsf{z} \urcorner = \langle 0 \rangle$ $\quad \ulcorner \mathsf{s} \urcorner = \langle 1 \rangle$ $\quad \ulcorner \pi_i^n \urcorner = \langle 2, n, i \rangle$
- $\ulcorner f \urcorner = \langle 3, \ulcorner g \urcorner, \ulcorner h_1 \urcorner, \dots, \ulcorner h_m \urcorner \rangle$ if f is obtained by composing g and $h_1, \dots, h_m$
- $\ulcorner f \urcorner = \langle 4, \ulcorner g \urcorner, \ulcorner h \urcorner \rangle$ if f is obtained by primitive recursion from g and h
- $\ulcorner f \urcorner = \langle 5, \ulcorner g \urcorner \rangle$ if f is obtained by minimizing g

Kleene's Normal Form Theorem

$\exists$ primitive recursive function u $\forall n \geqslant 1$ $\exists$ primitive recursive predicate T_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\quad \varphi(x_1, \dots, x_n) \simeq u((\mu y)\, T_n(\ulcorner \varphi \urcorner, x_1, \dots, x_n, y))$

Corollary

partial recursiveness and recursiveness coincide for total functions

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course−of−values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s−m−n theorem, total recursive functions, undecidability, while programs, . . .

Part II: Combinatory Logic and Lambda Calculus

α−equivalence, abstraction, arithmetization, β−reduction, CL−representability, combinators, combinatorial completeness, Church numerals, Church−Rosser theorem, Curry−Howard isomorphism, de Bruijn notation, η−reduction, fixed point theorem, intuitionistic propositional logic, λ−definability, normalization theorem, termination, typing, undecidability, Z property, . . .

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Outline

Example

$$0 + y = \pi^1_1(y)$$
$$(x+1) + y = \mathsf{s}(\pi^3_1(x+y, x, y))$$

$1 + 1 = 2$

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$1 + 1 = 2$:

- $\langle \ulcorner + \urcorner, \langle 1, 1 \rangle, 2 \rangle$
  - $\langle \ulcorner + \urcorner, \langle 0, 1 \rangle, 1 \rangle$
    - $\langle \ulcorner \pi^1_1 \urcorner, \langle 1 \rangle, 1 \rangle$
  - $\langle \ulcorner \mathsf{s} \circ \pi^3_1 \urcorner, \langle 1, 0, 1 \rangle, 2 \rangle$
    - $\langle \ulcorner \pi^3_1 \urcorner, \langle 1, 0, 1 \rangle, 1 \rangle$
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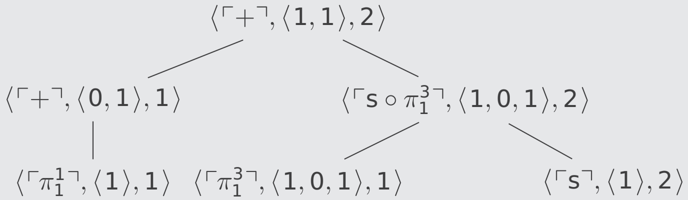
$\langle \ulcorner f \urcorner, \langle x_1, \ldots, x_n \rangle, z \rangle$ represents $f(x_1, \ldots, x_n) = z$

Example

$$0 + y = \pi_1^1(y)$$

$$(x+1) + y = \mathsf{s}(\pi_1^3(x+y, x, y))$$

$1 + 1 = 2$:



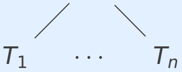
$\langle \ulcorner f \urcorner, \langle x_1, \dots, x_n \rangle, z \rangle$ represents $f(x_1, \dots, x_n) = z$

Notation

$\mathrm{triple}(x) = (\mathrm{seq}(x) \wedge \mathrm{len}(x) = 3)$

Key Idea

- encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$



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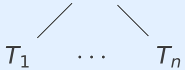

- dependencies between node $\langle x, y, z \rangle$ and root nodes of its children $T_1, \ldots, T_n$ can be checked by primitive recursive predicate

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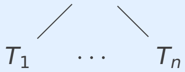
- encode computation tree $T = \langle x, y, z \rangle$ with children $T_1 \; \cdots \; T_n$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$
- dependencies between node $\langle x, y, z \rangle$ and root nodes of its children $T_1, \dots, T_n$ can be checked by primitive recursive predicate D

Definitions

- $\mathsf{T}(x) \iff \mathsf{D}(x) \wedge \left[\, \mathrm{len}(x) > 1 \iff (\forall\, i \leqslant \mathrm{len}(x) \dot{-} 2)\; \mathsf{T}((x)_{i+2}) \,\right]$

Key Idea

- encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$



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- $\mathrm{T}_n(e, \vec{x}, y) \iff \mathrm{T}(y) \wedge (y)_{1,1} = e \wedge (y)_{1,2} = \langle \vec{x} \rangle$

Key Idea

- encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$

$$\begin{array}{ccc} & \langle x, y, z \rangle & \\ \swarrow & & \searrow \\ T_1 & \cdots & T_n \end{array}$$

- dependencies between node $\langle x, y, z \rangle$ and root nodes of its children $T_1, \dots, T_n$ can be checked by primitive recursive predicate D

Definitions

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- encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$

  $T_1 \quad \cdots \quad T_n$

- dependencies between node $\langle x, y, z \rangle$ and root nodes of its children $T_1, \dots, T_n$ can be checked by primitive recursive predicate D

Definitions

- $\mathsf{T}(x) \iff \mathsf{D}(x) \wedge \big[\, \mathrm{len}(x) > 1 \iff (\forall\, i \leqslant \mathrm{len}(x) \mathbin{\dot-} 2)\, \mathsf{T}((x)_{i+2}) \,\big]$
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- $\mathsf{u}(x) = (x)_{1,3}$

Theorem

$\varphi(x_1, \dots, x_n) \simeq \mathsf{u}((\mu\, y)\, \mathsf{T}_n(\ulcorner \varphi \urcorner, x_1, \dots, x_n, y))$

Definition of D (cont'd)

$$\text{initial}(x) \iff \text{seq}(x) \wedge \text{len}(x) = 1 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge \text{seq}((x)_{1,2}) \wedge \Big[\, \text{zero}((x)_1) \vee \text{successor}((x)_1) \vee \text{projection}((x)_1) \,\Big]$$

Definition of D (cont'd)

- zero $\langle\langle\langle 0\rangle, \langle x\rangle, 0\rangle\rangle$

$$\begin{aligned}\mathrm{initial}(x) \iff{} & \mathrm{seq}(x) \wedge \mathrm{len}(x) = 1 \wedge \mathrm{triple}((x)_1) \wedge \mathrm{seq}((x)_{1,1}) \wedge \mathrm{seq}((x)_{1,2}) \wedge \\ & \big[\, \mathrm{zero}((x)_1) \vee \mathrm{successor}((x)_1) \vee \mathrm{projection}((x)_1) \,\big]\end{aligned}$$

with

$$\mathrm{zero}(x) \iff (x)_1 = \langle 0\rangle \wedge \mathrm{len}((x)_2) = 1 \wedge (x)_3 = 0$$

Definition of D (cont'd)

- zero $\langle\langle\langle 0\rangle, \langle x\rangle, 0\rangle\rangle$
- successor $\langle\langle\langle 1\rangle, \langle x\rangle, x+1\rangle\rangle$

$$\text{initial}(x) \iff \text{seq}(x) \wedge \text{len}(x) = 1 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge \text{seq}((x)_{1,2}) \wedge \Big[\, \text{zero}((x)_1) \vee \text{successor}((x)_1) \vee \text{projection}((x)_1) \,\Big]$$

with

$$\text{zero}(x) \iff (x)_1 = \langle 0\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = 0$$

$$\text{successor}(x) \iff (x)_1 = \langle 1\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = \text{s}((x)_{2,1})$$

Definition of D (cont'd)

- zero $\langle\langle\langle 0\rangle, \langle x\rangle, 0\rangle\rangle$
- successor $\langle\langle\langle 1\rangle, \langle x\rangle, x+1\rangle\rangle$
- projection $\langle\langle\langle 2, n, i\rangle, \langle x_1, \ldots, x_n\rangle, x_i\rangle\rangle$

$$\begin{aligned} \text{initial}(x) \iff\ & \text{seq}(x) \wedge \text{len}(x) = 1 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge \text{seq}((x)_{1,2}) \wedge \\ & \big[\, \text{zero}((x)_1) \vee \text{successor}((x)_1) \vee \text{projection}((x)_1) \,\big] \end{aligned}$$

with

$$\begin{aligned} \text{zero}(x) &\iff (x)_1 = \langle 0\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = 0 \\ \text{successor}(x) &\iff (x)_1 = \langle 1\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = \text{s}((x)_{2,1}) \\ \text{projection}(x) &\iff \text{len}((x)_1) = 3 \wedge (x)_{1,1} = 2 \wedge (x)_{1,2} = \text{len}((x)_2) \wedge (x)_{1,3} \geqslant 1 \wedge \\ & \qquad (x)_{1,2} \geqslant (x)_{1,3} \wedge (x)_3 = (x)_{2,(x)_{1,3}} \end{aligned}$$

Definition of D (cont'd)

- composition $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$

$$\langle \langle \ulcorner f \urcorner, \langle x_1, \ldots, x_n \rangle, z \rangle, T_1, \ldots, T_m, T_{m+1} \rangle$$

  with

$$\ulcorner f \urcorner = \langle 3, \ulcorner g \urcorner, \ulcorner h_1 \urcorner, \ldots, \ulcorner h_m \urcorner \rangle$$

$$T_1 = \langle \langle \ulcorner h_1 \urcorner, \langle x_1, \ldots, x_n \rangle, z_1 \rangle, \ldots \rangle \quad \cdots \quad T_m = \langle \langle \ulcorner h_m \urcorner, \langle x_1, \ldots, x_n \rangle, z_m \rangle, \ldots \rangle$$

$$T_{m+1} = \langle \langle \ulcorner g \urcorner, \langle z_1, \ldots, z_m \rangle, z \rangle, \ldots \rangle$$

Definition of D (cont'd)

- composition $f(x_1,\dots,x_n) = g(h_1(x_1,\dots,x_n),\dots,h_m(x_1,\dots,x_n))$

$$\langle\langle\ulcorner f\urcorner,\langle x_1,\dots,x_n\rangle,z\rangle,T_1,\dots,T_m,T_{m+1}\rangle$$

with

$$\begin{aligned}\ulcorner f\urcorner &= \langle 3,\ulcorner g\urcorner,\ulcorner h_1\urcorner,\dots,\ulcorner h_m\urcorner\rangle\\ T_1 &= \langle\langle\ulcorner h_1\urcorner,\langle x_1,\dots,x_n\rangle,z_1\rangle,\dots\rangle \quad\cdots\quad T_m = \langle\langle\ulcorner h_m\urcorner,\langle x_1,\dots,x_n\rangle,z_m\rangle,\dots\rangle\\ T_{m+1} &= \langle\langle\ulcorner g\urcorner,\langle z_1,\dots,z_m\rangle,z\rangle,\dots\rangle\end{aligned}$$

$$\begin{aligned}\text{composition}(x) \iff{}& \text{seq}(x) \wedge \text{len}(x) \geqslant 3 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge (x)_{1,1,1} = 3 \wedge\\ & \text{len}((x)_{1,1}) = \text{len}(x) \wedge (\forall\, i \leqslant \text{len}(x) \dot{-} 3)\,\big[\,(x)_{i+2,1,1} = (x)_{1,1,i+3} \wedge\\ & (x)_{i+2,1,2} = (x)_{1,2} \wedge (x)_{i+2,1,3} = (x)_{\text{len}(x),1,2,i+1}\,\big] \wedge\\ & (x)_{\text{len}(x),1,1} = (x)_{1,1,2} \wedge (x)_{\text{len}(x),1,3} = (x)_{1,3}\end{aligned}$$

Definition of D (cont'd)

- primitive recursion $\quad f(0,\vec{y}) = g(\vec{y}) \qquad f(x+1,\vec{y}) = h(f(x,\vec{y}),x,\vec{y})$

$$\langle\langle\ulcorner f\urcorner,\langle 0,y_1,\dots,y_n\rangle,z\rangle,\langle\langle\ulcorner g\urcorner,\langle y_1,\dots,y_n\rangle,z\rangle,\dots\rangle\rangle \quad \langle\langle\ulcorner f\urcorner,\langle x+1,y_1,\dots,y_n\rangle,z\rangle,T_1,T_2\rangle$$

with

$$\ulcorner f\urcorner = \langle 4,\ulcorner g\urcorner,\ulcorner h\urcorner\rangle$$

$$T_1 = \langle\langle\ulcorner f\urcorner,\langle x,y_1,\dots,y_n\rangle,z_1\rangle,\cdots\rangle \qquad T_2 = \langle\langle\ulcorner h\urcorner,\langle z_1,x,y_1,\dots,y_n\rangle,z\rangle,\cdots\rangle$$

Definition of D (cont'd)

- primitive recursion $f(0,\vec{y}) = g(\vec{y})$ $f(x+1,\vec{y}) = h(f(x,\vec{y}),x,\vec{y})$

$$\langle\langle\ulcorner f\urcorner,\langle 0,y_1,\ldots,y_n\rangle,z\rangle,\langle\langle\ulcorner g\urcorner,\langle y_1,\ldots,y_n\rangle,z\rangle,\ldots\rangle\rangle \quad \langle\langle\ulcorner f\urcorner,\langle x+1,y_1,\ldots,y_n\rangle,z\rangle,T_1,T_2\rangle$$

with

$$\ulcorner f\urcorner = \langle 4,\ulcorner g\urcorner,\ulcorner h\urcorner\rangle$$

$$T_1 = \langle\langle\ulcorner f\urcorner,\langle x,y_1,\ldots,y_n\rangle,z_1\rangle,\cdots\rangle \qquad T_2 = \langle\langle\ulcorner h\urcorner,\langle z_1,x,y_1,\ldots,y_n\rangle,z\rangle,\cdots\rangle$$

$$\begin{aligned}\text{recursion}(x) \iff{}& \text{seq}(x) \land \text{triple}((x)_1) \land \text{triple}((x)_{1,1}) \land (x)_{1,1,1} = 4 \land \text{seq}((x)_{1,2}) \land \\ & \big[(x)_{1,2,1} = 0 \implies \text{len}(x) = 2 \land (x)_{1,1,2} = (x)_{2,1,1} \land \\ & (x)_{1,2} = \langle 0\rangle ; (x)_{2,1,2} \land (x)_{1,3} = (x)_{2,1,3}\big] \land \big[(x)_{1,2,1} > 0 \implies \\ & \text{len}(x) = 3 \land (x)_{2,1,1} = (x)_{1,1} \land (x)_{2,1,2,1} = \text{p}((x)_{1,2,1}) \land \\ & (\forall\, i \leqslant \text{len}((x)_{1,2}) \dot{-} 2)\,\big[(x)_{1,2,i+2} = (x)_{2,1,2,i+2}\big] \land (x)_{3,1,1} = (x)_{1,1,3} \land \\ & (x)_{3,1,2} = \langle (x)_{2,1,3}\rangle ; (x)_{2,1,2} \land (x)_{3,1,3} = (x)_{1,3}\big]\end{aligned}$$

Definition of D (cont'd)

- minimization $f(y_1, \ldots, y_n) = (\mu\, x)\,(g(x, y_1, \ldots, y_n) = 0)$

  minimization(x) $\iff$ $\cdots$ (homework exercise)

Definition of D (cont'd)

- minimization $f(y_1, \ldots, y_n) = (\mu\, x)\, (g(x, y_1, \ldots, y_n) = 0)$

  minimization$(x) \iff \cdots$ (homework exercise)
- $\mathrm{D}(x) \iff$ initial(x) $\vee$ composition(x) $\vee$ recursion(x) $\vee$ minimization(x)

Definition of D (cont'd)

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Kleene's Normal Form Theorem

$\exists$ primitive recursive function u $\;\forall\, n \geqslant 1\;$ $\exists$ primitive recursive predicate T_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$

$$\varphi(x_1, \ldots, x_n) \simeq \mathrm{u}((\mu\, y)\, \mathrm{T}_n(\ulcorner \varphi \urcorner, x_1, \ldots, x_n, y))$$

Corollary

$\exists$ primitive recursive function u $\forall n \geqslant 1$ $\exists$ primitive recursive function t_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\quad \varphi(x_1,\dots,x_n) \simeq \mathrm{u}((\mu\, y)\,(\mathrm{t}_n(\ulcorner\varphi\urcorner, x_1,\dots,x_n,y) = 0))$

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Proof

define $\mathsf{t}_n = \chi_{\neg \mathsf{T}_n}$

Corollary

$\exists$ primitive recursive function u $\forall\, n \geqslant 1$ $\exists$ primitive recursive function t_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\quad \varphi(x_1,\dots,x_n) \simeq \mathrm{u}((\mu\, y)\; (\mathrm{t}_n(\ulcorner\varphi\urcorner, x_1,\dots,x_n,y) = 0))$

Proof

define $\mathrm{t}_n = \chi_{\neg \mathrm{T}_n}$

Definition

$\varphi_e^n(x_1,\dots,x_n) = \mathrm{u}((\mu\, y)\; (\mathrm{t}_n(e, x_1,\dots,x_n,y) = 0))$

Corollary

$\exists$ primitive recursive function u $\forall\, n \geqslant 1$ $\exists$ primitive recursive function t_n
$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\varphi(x_1,\dots,x_n) \simeq \mathrm{u}((\mu\, y)\,(\mathrm{t}_n(\ulcorner\varphi\urcorner, x_1,\dots,x_n,y) = 0))$

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Definition

$\varphi_e^n(x_1,\dots,x_n) = \mathrm{u}((\mu\, y)\,(\mathrm{t}_n(e, x_1,\dots,x_n,y) = 0))$

Lemma

- $\forall\, e \geqslant 0$ $\forall\, n \geqslant 1$ φ_e^n is partial recursive

Corollary

$\exists$ primitive recursive function u $\forall\, n \geqslant 1$ $\exists$ primitive recursive function t_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\quad \varphi(x_1,\dots,x_n) \simeq \mathrm{u}((\mu\, y)\,(\mathrm{t}_n(\ulcorner\varphi\urcorner, x_1,\dots,x_n,y) = 0))$

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$$\varphi_e^n(x_1,\dots,x_n) = \mathrm{u}((\mu\, y)\,(\mathrm{t}_n(e,x_1,\dots,x_n,y) = 0))$$

Lemma

- $\forall\, e \geqslant 0$ $\forall\, n \geqslant 1$ φ_e^n is partial recursive
- e is no index $\implies$ φ_e^n is nowhere defined

Corollary

$\exists$ primitive recursive function u $\forall\, n \geqslant 1$ $\exists$ primitive recursive function t_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\quad \varphi(x_1,\dots,x_n) \simeq \mathsf{u}((\mu\, y)\ (\mathsf{t}_n(\ulcorner\varphi\urcorner, x_1,\dots,x_n,y) = 0))$

Proof

define $\mathsf{t}_n = \chi_{\neg\mathsf{T}_n}$

Definition

$$\varphi_e^n(x_1,\dots,x_n) = \mathsf{u}((\mu\, y)\ (\mathsf{t}_n(e, x_1,\dots,x_n,y) = 0))$$

Lemma

- $\forall\, e \geqslant 0 \ \ \forall\, n \geqslant 1 \ \ \varphi_e^n$ is partial recursive
- e is no index $\implies$ φ_e^n is nowhere defined
- $\varphi_0^n, \varphi_1^n, \varphi_2^n, \dots$ is enumeration of all n-ary partial recursive functions

Outline

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is **decidable** if χ_P is recursive

Definition

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Theorem

following problem is undecidable:

instance: natural number x

question: is $\varphi_x(x)$ defined ?

Proof (by contradiction)

- suppose $f(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{if } \varphi_x(x)\uparrow \end{cases}$ is recursive

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- $g(e)\downarrow$

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- $g(e)\downarrow \iff f(e) = 0 \iff \varphi_e(e)\uparrow$ ↯

Outline

Example

- $x + y$ is partial recursive $\implies$ $\exists\, e$ such that $x + y = \varphi_e^2(x, y)$

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Kleene's s–m–n or Parameterization Theorem

$\forall\, m,n \geqslant 1$ $\exists$ primitive recursive function $s_n^m \colon \mathbb{N}^{m+1} \to \mathbb{N}$ $\forall\, e \in \mathbb{N}$

$$\varphi_e^{m+n}(x_1,\ldots,x_m,y_1,\ldots,y_n) \simeq \varphi^n_{s_n^m(e,x_1,\ldots,x_m)}(y_1,\ldots,y_n)$$

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$$f(0) = \langle 3, \langle 0 \rangle, \langle 2, n, 1 \rangle \rangle \qquad f(x+1) = \ulcorner s \circ c_x \urcorner = \langle 3, \langle 1 \rangle, f(x) \rangle$$

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- $\mathrm{s}_n^m(e, x_1, \dots, x_m) = \langle 3, e, f(x_1), \dots, f(x_m), \langle 2, n, 1 \rangle, \dots, \langle 2, n, n \rangle \rangle$

Outline

Kleene's Fixed Point Theorem

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construct partial recursive function that evaluates to its own index

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Outline

Important Concepts

- decidable predicate
- φ_e
- φ_e^n
- Kleene's fixed point theorem
- Kleene's normal form theorem
- Kleene's s–m–n theorem
- s_n^m
- T_n
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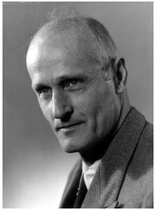

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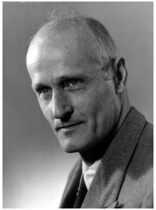

homework for November 6