universität innsbruck

WS 2023 lecture 6

Computability Theory

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Outline

1. **Summary of Previous Lecture**
2. **Recursive and Recursive Enumerable Sets**
3. **Diophantine Sets**
4. **Fibonacci Numbers**
5. **Summary**

Definition

$$\varphi_e^n(x_1,\dots,x_n) = \mathrm{u}((\mu\, y)\ (\mathrm{t}_n(e, x_1,\dots,x_n,y) = 0))$$

Lemma

$\varphi_0^n,\ \varphi_1^n,\ \varphi_2^n,\ \dots$ is computable enumeration of all n-ary partial recursive functions

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is decidable if χ_P is recursive

Theorem

following problem is undecidable:

instance: natural number x

question: is $\varphi_x(x)$ defined ?

Kleene's s–m–n or Parameterization Theorem

$\forall\, m, n \geqslant 1 \quad \exists$ primitive recursive function $s^m_n : \mathbb{N}^{m+1} \to \mathbb{N} \quad \forall\, e \in \mathbb{N}$

$$\varphi^{m+n}_e(x_1, \dots, x_m, y_1, \dots, y_n) \simeq \varphi^n_{s^m_n(e, x_1, \dots, x_m)}(y_1, \dots, y_n)$$

Kleene's Fixed Point Theorem

$\forall$ recursive function $f : \mathbb{N} \to \mathbb{N} \quad \exists\, e \in \mathbb{N}$ such that $\varphi_e(x) \simeq \varphi_{f(e)}(x)$

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability, while programs, ...

Part II: Combinatory Logic and Lambda Calculus

α–equivalence, abstraction, arithmetization, β–reduction, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, de Bruijn notation, η–reduction, fixed point theorem, intuitionistic propositional logic, λ–definability, normalization theorem, termination, typing, undecidability, Z property, ...

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Definitions

- set $A \subseteq \mathbb{N}$ is recursive if its characteristic function $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ is recursive

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- disjoint sets $A, B \subseteq \mathbb{N}$ are recursively separable if $\exists$ recursive function $f \colon \mathbb{N} \to \{0, 1\}$ such that

$$x \in A \implies f(x) = 0 \qquad\qquad x \in B \implies f(x) = 1$$

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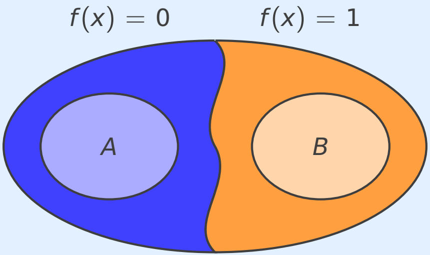

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sets $A = \{x \mid \varphi_x(x) = 0\}$ and $B = \{x \mid \varphi_x(x) = 1\}$ are recursively inseparable

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- suppose $\exists$ recursive function $f \colon \mathbb{N} \to \{0, 1\}$ separating A and B

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- $g(x) = 1 \dot{-} f(x)$ is recursive

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$$f(e) = 0 \implies \varphi_e(e) = 1 \implies e \in B$$
$$f(e) = 1 \implies \varphi_e(e) = 0 \implies e \in A$$

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f(e) = 1 & \Longrightarrow & \varphi_e(e) = 0 & \Longrightarrow & e \in A & \Longrightarrow & f(e) = 0
\end{array}
$$

↯

Definition

set $A \subseteq \mathbb{N}$ is index set if

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for all $d, e \in \mathbb{N}$

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Rice's Theorem

non-trivial index sets are not recursive

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Proof

- let A be non-trivial index set

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- let A be non-trivial index set and let $d \in A$ and $e \notin A$

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Proof

- let A be non-trivial index set and let $d \in A$ and $e \notin A$
- suppose A is recursive

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- let A be non-trivial index set and let $d \in A$ and $e \notin A$
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- function f defined by $f(x) = \begin{cases} e & \text{if } x \in A \\ d & \text{if } x \notin A \end{cases}$ is recursive

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  $a \in A$

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Proof

- let A be non-trivial **index set** and let $d \in A$ and $e \notin A$
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$$a \in A \implies f(a) \in A$$

$$a \notin A \implies f(a) \notin A$$

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$$a \in A \implies f(a) \in A \implies e \in A$$

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$$\begin{array}{lllll} a \in A & \implies & f(a) \in A & \implies & e \in A \\ a \notin A & \implies & f(a) \notin A & \implies & d \notin A \end{array} \quad ↯$$

Definition

set $A \subseteq \mathbb{N}$ is **recursively enumerable** if $A = \varnothing$ or A is range of unary recursive function

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other terminology: semi-decidable computably enumerable

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Lemma

set A is recursive if and only if A and $\mathbb{N} \setminus A$ are recursively enumerable

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Lemma

set A is recursive if and only if A and $\mathbb{N} \setminus A$ are recursively enumerable

Theorem

following statements are equivalent for any set $A \subseteq \mathbb{N}$:

1. A is recursively enumerable
2. A is range of unary partial recursive function
3. A is domain of unary partial recursive function

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Proof ① $\Longrightarrow$ ③

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Proof 1 $\Longrightarrow$ 3

- $A = \varnothing$ or A is range of unary recursive function

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Proof 1 $\Longrightarrow$ 3

- $A = \varnothing$ or A is range of unary recursive function
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$

1. A is recursively enumerable
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Proof 1 $\implies$ 3

- $A = \varnothing$ or A is range of unary recursive function f
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$
- define unary function $g(x) = (\mu\, y)\,(f(y) = x)$

1. A is recursively enumerable
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Proof ① $\Longrightarrow$ ③

- $A = \varnothing$ or A is range of unary recursive function f
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$
- define unary function $g(x) = (\mu\, y)\,(f(y) = x)$
- g is partial recursive

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Proof 1 $\Longrightarrow$ 3

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- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$
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- g is partial recursive
- domain of g is A

1. A is recursively enumerable
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Proof ③ $\Longrightarrow$ ②

1. A is recursively enumerable
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Proof 3 $\Longrightarrow$ 2

- suppose A is domain of unary partial recursive function φ_e

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3. A is domain of unary partial recursive function

Proof ③ $\Longrightarrow$ ②

- suppose A is domain of unary partial recursive function φ_e
- define unary function $f(x) = x + z(\varphi_e(x))$

1. A is recursively enumerable
2. A is range of unary partial recursive function
3. A is domain of unary partial recursive function

Proof (3) $\Longrightarrow$ (2)

- suppose A is domain of unary partial recursive function φ_e
- define unary function $f(x) = x + z(\varphi_e(x))$
- f is partial recursive

1. A is recursively enumerable
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Proof ③ $\Longrightarrow$ ②

- suppose A is domain of unary partial recursive function φ_e
- define unary function $f(x) = x + z(\varphi_e(x))$
- f is partial recursive
- $\varphi_e(x)\downarrow \iff f(x)\downarrow$

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Proof ③ $\Longrightarrow$ ②

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- $\varphi_e(x)\downarrow \iff f(x)\downarrow \implies f(x) = x$

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Proof 3 $\Longrightarrow$ 2

- suppose A is domain of unary partial recursive function φ_e
- define unary function $f(x) = x + z(\varphi_e(x))$
- f is partial recursive
- $\varphi_e(x){\downarrow} \iff f(x){\downarrow} \implies f(x) = x$
- range of f is A

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Proof 2 $\Longrightarrow$ 1

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Proof 2 $\Longrightarrow$ 1

- suppose A is range of unary partial recursive function φ_e
- if $A = \varnothing$ then A is recursively enumerable

1. A is recursively enumerable
2. A is range of unary partial recursive function
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Proof ② $\Longrightarrow$ ①

- suppose A is range of unary partial recursive function φ_e
- if $A = \varnothing$ then A is recursively enumerable suppose $A \neq \varnothing$ and let $a \in A$

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Proof 2 $\implies$ 1

- suppose A is range of unary partial recursive function φ_e
- if $A = \varnothing$ then A is recursively enumerable suppose $A \neq \varnothing$ and let $a \in A$
- define $f(x,s) = \begin{cases} \varphi_e(x) & \text{if } (\exists\, y < s)\, \mathrm{T}_1(e,x,y) \\ a & \text{otherwise} \end{cases}$ and $g(x) = f(\pi_1(x), \pi_2(x))$

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- f and g are recursive
- claim: range of g is A

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Proof (2) $\Longrightarrow$ (1)

- suppose A is range of unary partial recursive function φ_e
- if $A = \varnothing$ then A is recursively enumerable   suppose $A \neq \varnothing$ and let $a \in A$
- define $f(x,s) = \begin{cases} \varphi_e(x) & \text{if } (\exists\, y < s)\ \mathrm{T}_1(e,x,y) \\ a & \text{otherwise} \end{cases}$ and $g(x) = f(\pi_1(x), \pi_2(x))$
- f and g are recursive
- claim: range of g is A
- $z \in A \implies z = \varphi_e(x)$ for some x

1. A is recursively enumerable
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Proof 2 $\Longrightarrow$ 1

- suppose A is range of unary partial recursive function φ_e
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- f and g are recursive
- claim: range of g is A
- $z \in A \implies z = \varphi_e(x)$ for some $x \implies \mathrm{T}_1(e,x,y)$ for some y

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Proof (2) $\Longrightarrow$ (1)

- suppose A is range of unary partial recursive function φ_e
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- f and g are recursive
- claim: range of g is A
- $z \in A \implies z = \varphi_e(x)$ for some $x \implies \mathrm{T}_1(e,x,y)$ for some y
- $f(x,s) = z$ for any $s > y$

1. A is recursively enumerable
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Proof 2 $\Longrightarrow$ 1

- suppose A is range of unary partial recursive function φ_e
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- f and g are recursive
- claim: range of g is A
- $z \in A \implies z = \varphi_e(x)$ for some $x \implies \mathrm{T}_1(e,x,y)$ for some y
- $f(x,s) = z$ for any $s > y \implies g(\pi(x,s)) = z$ for any $s > y$

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every non-empty recursively enumerable set $A \subseteq \mathbb{N}$ is range of primitive recursive function

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Proof

- A is range of unary recursive function f

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Proof

- A is range of unary recursive function f
- let e be index of f and let a be arbitrary element of A

Lemma

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Proof

- A is range of unary recursive function f
- let e be index of f and let a be arbitrary element of A
- function

$$g(x) = \begin{cases} \mathrm{u}((x)_1) & \text{if } \mathrm{T}_1(e,(x)_0,(x)_1) \\ a & \text{otherwise} \end{cases}$$

is primitive recursive

Lemma

every non-empty recursively enumerable set $A \subseteq \mathbb{N}$ is range of primitive recursive function

Proof

- A is range of unary recursive function f
- let e be index of f and let a be arbitrary element of A
- function

$$g(x) = \begin{cases} \mathrm{u}((x)_1) & \text{if } \mathrm{T}_1(e,(x)_0,(x)_1) \\ a & \text{otherwise} \end{cases}$$

  is primitive recursive
- range of g is A

Outline

Definition

set $A \subseteq \mathbb{N}$ is diophantine if $\exists$ polynomial $P(x, y_1, \ldots, y_n)$ with integer coefficients such that

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Examples

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Lemma

diophantine sets are recursively enumerable

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Proof

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Theorem (Matiyasevich 1970)

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Corollary (MRDP Theorem)

Hilbert's 10th problem is unsolvable

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Example

polynomial $P(a, b, \ldots, z)$:

$$
\begin{aligned}
(k+2)\Big(1 &- (wz+h+j-q)^2 - \big((gk+2g+k+1)(h+j)+h-z\big)^2 \\
&- (2n+p+q+z-e)^2 - \big((a^2-1)y^2+1-x^2\big)^2 \\
&- \big(16(k+1)^3(k+2)(n+1)^2+1-f^2\big)^2 - (n+l+v-y)^2 \\
&- \big(16r^2y^4(a^2-1)+1-u^2\big)^2 - (e^3(e+2)(a+1)^2+1-o^2)^2 \\
&- \big(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2\big)^2 \\
&- \big((a^2-1)l^2+1-m^2\big)^2 - (ai+k+1-l-i)^2 \\
&- \big(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m\big)^2 \\
&- \big(z+pl(a-p)+t(2ap-p^2-1)-pm\big)^2 \\
&- \big(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x\big)^2\Big)
\end{aligned}
$$

Example (Jones, Sato, Wada, Wiens 1976)

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$$(k+2)\Big(1-(wz+h+j-q)^2-\big((gk+2g+k+1)(h+j)+h-z\big)^2$$
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generates all prime numbers

Outline

Definition (Fibonacci numbers)

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

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Theorem (Jones 1975)

$P(x,y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of Fibonacci numbers

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$$= -(F_i^2 - F_i F_{i-1} - F_{i-1}^2) = -(-1)^{i-1} = (-1)^i$$

Lemma ❷

$y^2 - yx - x^2 = 1 \ \wedge \ x, y \geqslant 0 \quad \Longrightarrow \quad x = F_{2i}$ and $y = F_{2i+1}$ for some $i \geqslant 0$

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Proof

induction on x

- $x = 0 \implies y = 1 \implies i = 0$
- $x > 0 \implies yx + x^2 > y$

Lemma ❷

$y^2 - yx - x^2 = 1 \wedge x, y \geqslant 0 \implies x = F_{2i}$ and $y = F_{2i+1}$ for some $i \geqslant 0$

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  $x = a + b$ and $y = b + x \implies x = F_{2(i+1)}$ and $y = F_{2(i+1)+1}$

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$$y^2 - yx - x^2 = -1 \;\wedge\; x \geqslant 0 \;\wedge\; y > 0 \quad\Longrightarrow\quad x = F_{2i+1} \text{ and } y = F_{2i+2} \text{ for some } i \geqslant 0$$

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case analysis

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Corollary ❶

$\{x \geqslant 0 \mid y^2 - yx - x^2 = \pm 1 \text{ for some } y > 0\}$ is set of Fibonacci numbers

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Corollary

set of Fibonacci numbers is diophantine

Lemma ❹

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  $$4\,(y^2 - yx - x^2) = (2y - x)^2 - 5\,x^2$$

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  $$(2y - x)^2 - 5x^2 \neq 0$$

Lemma ④

$$y > 0 \wedge x \geqslant 0 \quad \Longrightarrow \quad y^2 - yx - x^2 \neq 0$$

Proof

case analysis

- $x = 0 \quad \Longrightarrow \quad y^2 - yx - x^2 = y^2 > 0$
- $x > 0$

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Theorem

set of Fibonacci numbers equals non-negative values of $P(x,y) = x(2 - (y^2 - yx - x^2)^2)$

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    $(y^2-yx-x^2)^2 = 1 \implies z = x \implies z$ is Fibonacci number by corollary 2

Theorem (Jones 1975)

$P(x,y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of Fibonacci numbers

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Remark

$P(2,2) = -28$

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Remark

$P(2,2) = -28$

Theorem

there exists no polynomial $Q(x_1,\dots,x_n)$ such that

$$\{Q(x_1,\dots,x_n) \mid x_1,\dots,x_n \geqslant 0\}$$

is set of Fibonacci numbers

Outline

1. **Summary of Previous Lecture**
2. **Recursive and Recursive Enumerable Sets**
3. **Diophantine Sets**
4. **Fibonacci Numbers**
5. **Summary**

Important Concepts

- diophantine set
- Fibonacci numbers
- index set
- recursive set
- recursive enumerable set
- recursive separable sets

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homework for November 13