universität innsbruck

WS 2023 lecture 6

Computability Theory

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Outline

Definition

$\varphi_e^n(x_1, \ldots, x_n) = \mathrm{u}((\mu\, y)\ (\mathrm{t}_n(e, x_1, \ldots, x_n, y) = 0))$

Lemma

$\varphi_0^n,\ \varphi_1^n,\ \varphi_2^n,\ \ldots$ is computable enumeration of all n-ary partial recursive functions

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is decidable if χ_P is recursive

Theorem

following problem is undecidable:

instance: natural number x
question: is $\varphi_x(x)$ defined ?

Kleene's s–m–n or Parameterization Theorem

$\forall\, m, n \geqslant 1 \quad \exists$ primitive recursive function $s_n^m\colon \mathbb{N}^{m+1} \to \mathbb{N} \quad \forall\, e \in \mathbb{N}$

$$\varphi_e^{m+n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \simeq \varphi^n_{s_n^m(e, x_1, \ldots, x_m)}(y_1, \ldots, y_n)$$

Kleene's Fixed Point Theorem

$\forall$ recursive function $f\colon \mathbb{N} \to \mathbb{N} \quad \exists\, e \in \mathbb{N}$ such that $\varphi_e(x) \simeq \varphi_{f(e)}(x)$

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability, while programs, ...

Part II: Combinatory Logic and Lambda Calculus

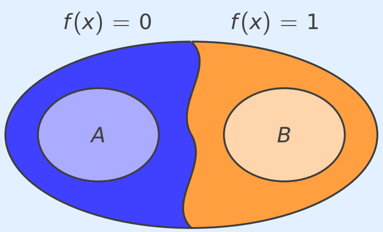
α–equivalence, abstraction, arithmetization, β–reduction, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, de Bruijn notation, η–reduction, fixed point theorem, intuitionistic propositional logic, λ–definability, normalization theorem, termination, typing, undecidability, Z property, ...

Outline

Definitions

- set $A \subseteq \mathbb{N}$ is recursive if its characteristic function $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ is recursive
- disjoint sets $A, B \subseteq \mathbb{N}$ are recursively separable if $\exists$ recursive function $f\colon \mathbb{N} \to \{0,1\}$ such that

$$x \in A \implies f(x) = 0 \qquad\qquad x \in B \implies f(x) = 1$$



Lemma

if A and B are recursively inseparable then A and B are not recursive

Theorem

sets $A = \{x \mid \varphi_x(x) = 0\}$ and $B = \{x \mid \varphi_x(x) = 1\}$ are recursively inseparable

Proof

- suppose $\exists$ recursive function $f\colon \mathbb{N} \to \{0,1\}$ separating A and B
- $g(x) = 1 \dot{-} f(x)$ is recursive
- $\exists\, e$ such that $g = \varphi_e$

$$f(e) = 0 \implies \varphi_e(e) = 1 \implies e \in B \implies f(e) = 1$$
$$f(e) = 1 \implies \varphi_e(e) = 0 \implies e \in A \implies f(e) = 0$$

↯

Definition

set $A \subseteq \mathbb{N}$ is index set if

$$d \in A \wedge \varphi_e \simeq \varphi_d \implies e \in A$$

for all $d, e \in \mathbb{N}$

Examples

- $\varnothing$ and $\mathbb{N}$ are (trivial) index sets
- $\{ e \mid \varphi_e$ is recursive function$\}$ is index set
- $\{\langle 0\rangle, \langle 1\rangle\} \cup \{\langle 2, n, i\rangle \mid 1 \leqslant i \leqslant n\}$ is no index set

Rice's Theorem

non-trivial index sets are not recursive

Rice's Theorem

non-trivial index sets are not recursive

Proof

- let A be non-trivial index set and let $d \in A$ and $e \notin A$
- suppose A is recursive
- function f defined by $f(x) = \begin{cases} e & \text{if } x \in A \\ d & \text{if } x \notin A \end{cases}$ is recursive
- fixed point theorem $\implies \exists\, a$ such that $\varphi_a \simeq \varphi_{f(a)}$

$$a \in A \implies f(a) \in A \implies e \in A$$
$$a \notin A \implies f(a) \notin A \implies d \notin A$$

↯

Definition

set $A \subseteq \mathbb{N}$ is recursively enumerable if $A = \varnothing$ or A is range of unary recursive function

Remark

other terminology: semi-decidable computably enumerable

Lemma

set A is recursive if and only if A and $\mathbb{N} \setminus A$ are recursively enumerable

Theorem

following statements are equivalent for any set $A \subseteq \mathbb{N}$:

1. A is recursively enumerable
2. A is range of unary partial recursive function
3. A is domain of unary partial recursive function

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2. A is range of unary partial recursive function
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Proof ❶ $\implies$ ❸

- $A = \varnothing$ or A is range of unary recursive function f
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$
- define unary function $g(x) = (\mu\, y)\,(f(y) = x)$
- g is partial recursive
- domain of g is A

1. A is recursively enumerable
2. A is range of unary partial recursive function
3. A is domain of unary partial recursive function

Proof ③ $\Longrightarrow$ ②

- suppose A is domain of unary partial recursive function φ_e
- define unary function $f(x) = x + \mathrm{z}(\varphi_e(x))$
- f is partial recursive
- $\varphi_e(x)\downarrow \iff f(x)\downarrow \implies f(x) = x$
- range of f is A

1. A is recursively enumerable
2. A is range of unary partial recursive function
3. A is domain of unary partial recursive function

Proof ② $\Longrightarrow$ ①

- suppose A is range of unary partial recursive function φ_e
- if $A = \varnothing$ then A is recursively enumerable suppose $A \neq \varnothing$ and let $a \in A$
- define $f(x,s) = \begin{cases} \varphi_e(x) & \text{if } (\exists\, y < s)\ \mathrm{T}_1(e,x,y) \\ a & \text{otherwise} \end{cases}$ and $g(x) = f(\pi_1(x), \pi_2(x))$
- f and g are recursive
- claim: range of g is A
- $z \in A \implies z = \varphi_e(x)$ for some $x \implies \mathrm{T}_1(e,x,y)$ for some y
- $f(x,s) = z$ for any $s > y \implies g(\pi(x,s)) = z$ for any $s > y$

Lemma

every non-empty recursively enumerable set $A \subseteq \mathbb{N}$ is range of primitive recursive function

Proof

- A is range of unary recursive function f
- let e be index of f and let a be arbitrary element of A
- function

$$g(x) = \begin{cases} \mathrm{u}((x)_1) & \text{if } \mathrm{T}_1(e,(x)_0,(x)_1) \\ a & \text{otherwise} \end{cases}$$

  is primitive recursive
- range of g is A

Outline

Definition

set $A \subseteq \mathbb{N}$ is diophantine if $\exists$ polynomial $P(x, y_1, \dots, y_n)$ with integer coefficients such that

$$x \in A \iff \exists\, y_1 \cdots \exists\, y_n\; P(x, y_1, \dots, y_n) = 0$$

Examples

- $\{x \mid x \text{ is even}\}$ $\quad P(x,y) = x - 2y$
- $\{x^2 \mid x \in \mathbb{N}\}$ $\quad P(x,y) = x - y^2$
- $\{x \geqslant 1 \mid x \text{ is composite}\}$ $\quad P(x,y,z) = x - (y+2)(z+2)$

Lemma

diophantine sets are recursively enumerable

Proof

- arbitrary diophantine set $A = \{x \mid \exists\, y_1 \cdots \exists\, y_n\, P(x, y_1, \dots, y_n) = 0\}$
- function $\varphi(x) = (\mu\, y)\, (P(x, (y)_1, \dots, (y)_n)^2 = 0)$ is partial recursive
- $x \in A \iff \varphi(x)\!\downarrow$

Theorem (Matiyasevich 1970)

recursively enumerable sets are diophantine

Corollary (MRDP Theorem)

Hilbert's 10th problem is unsolvable

Lemma

A is recursively enumerable $\iff$ $A = \{P(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{N} \text{ and } P(x_1, \dots, x_n) \geqslant 0\}$ for some polynomial $P(x_1, \dots, x_n)$ with integer coefficients

Proof

$\Longleftarrow$ $\quad x \in A \iff \exists\, x_1 \cdots \exists\, x_n \;\; P(x_1, \dots, x_n) - x = 0$

$\Longrightarrow$ $\quad \exists$ polynomial $Q(x, y_1, \dots, y_n)$ with integer coefficients such that

$$x \in A \iff \exists\, y_1 \cdots \exists\, y_n \;\; Q(x, y_1, \dots, y_n) = 0$$

define $P(x, y_1, \dots, y_n) = x - (x+1)\,(Q(x, y_1, \dots, y_n))^2$

$$\begin{aligned} &\{P(x, y_1, \dots, y_n) \mid x, y_1, \dots, y_n \in \mathbb{N} \text{ and } P(x, y_1, \dots, y_n) \geqslant 0\} \\ = &\{P(x, y_1, \dots, y_n) \mid x, y_1, \dots, y_n \in \mathbb{N} \text{ and } Q(x, y_1, \dots, y_n) = 0\} \\ = &\{x \mid x, y_1, \dots, y_n \in \mathbb{N} \text{ and } Q(x, y_1, \dots, y_n) = 0\} \; = \; A \end{aligned}$$

Example (Jones, Sato, Wada, Wiens 1976)

polynomial $P(a, b, \dots, z)$:

$$\begin{aligned} (k+2)\Big(1 &- (wz+h+j-q)^2 - \big((gk+2g+k+1)(h+j)+h-z\big)^2 \\ &- (2n+p+q+z-e)^2 - \big((a^2-1)y^2+1-x^2\big)^2 \\ &- \big(16(k+1)^3(k+2)(n+1)^2+1-f^2\big)^2 - (n+l+v-y)^2 \\ &- \big(16r^2y^4(a^2-1)+1-u^2\big)^2 - (e^3(e+2)(a+1)^2+1-o^2)^2 \\ &- \big(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2\big)^2 \\ &- \big((a^2-1)l^2+1-m^2\big)^2 - (ai+k+1-l-i)^2 \\ &- \big(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m\big)^2 \\ &- \big(z+pl(a-p)+t(2ap-p^2-1)-pm\big)^2 \\ &- \big(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x\big)^2\Big) \end{aligned}$$

generates all prime numbers

Outline

Definition (Fibonacci numbers)

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

Theorem (Jones 1975)

$P(x,y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of Fibonacci numbers

demo

Definition (Fibonacci numbers)

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

Lemma ❶

$F_{i+1}^2 - F_{i+1}F_i - F_i^2 = (-1)^i$ for all $i \geqslant 0$

Proof

induction on i

- $i = 0 \implies F_{i+1}^2 - F_{i+1}F_i - F_i^2 = 1 - 0 - 0 = 1 = (-1)^0$
- $i = 1 \implies F_{i+1}^2 - F_{i+1}F_i - F_i^2 = 1 - 1 - 1 = -1 = (-1)^1$
- $i > 1 \implies F_{i+1}^2 - F_{i+1}F_i - F_i^2 = (F_{i-1} + F_i)^2 - (F_{i-1} + F_i)F_i - F_i^2 = -F_i^2 + F_{i-1}^2 + F_iF_{i-1}$

$$= -(F_i^2 - F_iF_{i-1} - F_{i-1}^2) = -(-1)^{i-1} = (-1)^i$$

Lemma ❷

$y^2 - yx - x^2 = 1 \wedge x, y \geqslant 0 \implies x = F_{2i}$ and $y = F_{2i+1}$ for some $i \geqslant 0$

Proof

induction on x

- $x = 0 \implies y = 1 \implies i = 0$
- $x > 0 \implies yx + x^2 > y \implies 1 = y^2 - (yx + x^2) < y^2 - y \implies y \geqslant 2$

$(x+1)^2 = x^2 + 2x + 1 \leqslant x^2 + yx + 1 = y^2 \implies y > x$

$y^2 = yx + x^2 + 1 \leqslant yx + x^2 + x = yx + (x+1)x \leqslant yx + yx = 2yx \implies y \leqslant 2x$

let $a = 2x - y$ and $b = y - x \implies 0 \leqslant a < x$ and $0 < b$

$b^2 - ba - a^2 = (y-x)^2 - (y-x)(2x-y) - (2x-y)^2 = y^2 - yx - x^2 = 1$

$a = F_{2i}$ and $b = F_{2i+1}$ for some $i \geqslant 0$ by induction hypothesis

$x = a + b$ and $y = b + x \implies x = F_{2(i+1)}$ and $y = F_{2(i+1)+1}$

Lemma ❸

$y^2 - yx - x^2 = -1 \wedge x \geqslant 0 \wedge y > 0 \implies x = F_{2i+1}$ and $y = F_{2i+2}$ for some $i \geqslant 0$

Proof

case analysis

- $x \leqslant y$

  let $a = y - x$ and $b = x$

  $a \geqslant 0$ and $b \geqslant 0$

  $b^2 - ba - a^2 = x^2 - x(y-x) - (y-x)^2 = -(y^2 - yx - x^2) = 1$

  $a = F_{2i}$ and $b = F_{2i+1}$ for some $i \geqslant 0$ according to lemma ❷

  $x = F_{2i+1}$ and $y = a + x = F_{2i+2}$

- $y < x \implies yx = y^2 - x^2 + 1 \leqslant 0 \implies yx = 0 \implies y = 0$ ↯

Corollary ❶

$\{x \geqslant 0 \mid y^2 - yx - x^2 = \pm 1$ for some $y > 0\}$ is set of Fibonacci numbers

Corollary ❷

$\{x \geqslant 0 \mid (y^2 - yx - x^2)^2 - 1 = 0$ for some $y \geqslant 0\}$ is set of Fibonacci numbers

Corollary

set of Fibonacci numbers is diophantine

Lemma ❹

$y > 0 \wedge x \geqslant 0 \implies y^2 - yx - x^2 \neq 0$

Proof

case analysis

- $x = 0 \implies y^2 - yx - x^2 = y^2 > 0$
- $x > 0$

  $4(y^2 - yx - x^2) = (2y - x)^2 - 5x^2$

  $(2y - x)^2 - 5x^2 = 0 \implies \sqrt{5} = \dfrac{|2y - x|}{x}$ ↯

  $(2y - x)^2 - 5x^2 \neq 0 \implies y^2 - yx - x^2 \neq 0$

Theorem

set of Fibonacci numbers equals non-negative values of $P(x,y) = x(2 - (y^2 - yx - x^2)^2)$

Proof

two directions

- x is Fibonacci number $\implies x = F_i$ for some $i \geqslant 0$

  $y = F_{i+1} \implies (y^2 - yx - x^2)^2 = 1 \implies 2 - (y^2 - yx - x^2)^2 = 1 \implies P(x,y) = x$

- $z = P(x,y) > 0$ for some $x, y \geqslant 0 \implies z = x(2 - (y^2 - yx - x^2)^2)$

  two cases

  - $y = 0 \implies z = x(2 - x^4) \implies x = 1 \implies x$ is Fibonacci number
  - $y > 0 \implies y^2 - yx - x^2 \neq 0$ by lemma ❹ $\implies 0 < (y^2 - yx - x^2)^2 < 2$

    $(y^2 - yx - x^2)^2 = 1 \implies z = x \implies z$ is Fibonacci number by corollary ❷

Theorem (Jones 1975)

$P(x,y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of Fibonacci numbers

Proof

$2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x = x(2 - (y^2 - yx - x^2)^2)$

Remark

$P(2,2) = -28$

Theorem

there exists no polynomial $Q(x_1,\dots,x_n)$ such that

$$\{Q(x_1,\dots,x_n) \mid x_1,\dots,x_n \geqslant 0\}$$

is set of Fibonacci numbers

Outline

Important Concepts

- diophantine set
- Fibonacci numbers
- index set
- recursive set
- recursive enumerable set
- recursive separable sets

homework for November 13