WS 2023 lecture 7

Computability Theory

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Outline

Definitions

- set $A \subseteq \mathbb{N}$ is recursive if its characteristic function χ_A is recursive
- disjoint sets $A, B \subseteq \mathbb{N}$ are recursively separable if there exists $f\colon \mathbb{N} \to \{0,1\} \in \mathrm{R}$ such that

$$x \in A \implies f(x) = 0 \qquad x \in B \implies f(x) = 1$$

- set $A \subseteq \mathbb{N}$ is recursively enumerable if $A = \varnothing$ or A is range of unary recursive function
- set $A \subseteq \mathbb{N}$ is index set if $d \in A \wedge \varphi_e \simeq \varphi_d \implies e \in A$ for all $d, e \in \mathbb{N}$

Lemmata

- if A and B are recursively inseparable then A and B are not recursive
- set A is recursive if and only if A and $\mathbb{N} \setminus A$ are recursively enumerable

Rice's Theorem

non-trivial index sets are not recursive

Theorem

- sets $A = \{x \mid \varphi_x(x) = 0\}$ and $B = \{x \mid \varphi_x(x) = 1\}$ are recursively inseparable
- following statements are equivalent for any set $A \subseteq \mathbb{N}$:
  1. A is recursively enumerable
  2. A is range of unary partial recursive function
  3. A is domain of unary partial recursive function

Definition

set $A \subseteq \mathbb{N}$ is diophantine if $\exists$ polynomial $P(x, y_1, \dots, y_n)$ with integer coefficients such that

$$x \in A \iff \exists y_1 \cdots \exists y_n \;\; P(x, y_1, \dots, y_n) = 0$$

Lemma

A is diophantine $\iff$ $A = \{P(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{N} \text{ and } P(x_1, \dots, x_n) \geqslant 0\}$
for some polynomial $P(x_1, \dots, x_n)$ with integer coefficients

Lemma

diophantine sets are recursively enumerable

Theorem (Matiyasevich)

recursively enumerable sets are diophantine

Corollary (MRDP Theorem)

Hilbert's 10th problem is unsolvable

Theorem (Jones 1975)

- $P(x,y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of Fibonacci numbers
- there exists no polynomial $Q(x_1,\ldots,x_n)$ such that

$$\{Q(x_1,\ldots,x_n) \mid x_1,\ldots,x_n \geqslant 0\}$$

  is set of Fibonacci numbers

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability, while programs, ...

Part II: Combinatory Logic and Lambda Calculus

α–equivalence, abstraction, arithmetization, β–reduction, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, de Bruijn notation, η–reduction, fixed point theorem, intuitionistic propositional logic, λ–definability, normalization theorem, termination, typing, undecidability, Z property, ...

Outline

Definition

CL–terms are built from

- infinite set of variables $\mathcal{V} = \{x, y, z, \ldots\}$
- constants I K S
- application $(s \cdot t)$ for CL–terms s and t

Examples

y	S	$(\mathsf{I} \cdot \mathsf{K})$	$(x \cdot (\mathsf{S} \cdot y))$	$((\mathsf{S} \cdot \mathsf{I}) \cdot \mathsf{I})$	functional notation
y	S	IK	$x(\mathsf{S}y)$	SII	applicative notation

Notational Conventions

- juxtaposition for application
- association to the left to reduce number of parentheses

Definitions

- set $\mathcal{V}\mathrm{ar}(t)$ of variables of CL–term t is inductively defined:

$$\mathcal{V}\mathrm{ar}(t) = \begin{cases} \varnothing & \text{if } t \in \{\mathsf{I}, \mathsf{K}, \mathsf{S}\} \\ \{t\} & \text{if } t \in \mathcal{V} \\ \mathcal{V}\mathrm{ar}(t_1) \cup \mathcal{V}\mathrm{ar}(t_2) & \text{if } t = t_1 t_2 \end{cases}$$

- combinator is CL–term t such that $\mathcal{V}\mathrm{ar}(t) = \varnothing$

Definition

(weak) reduction is smallest relation $\to$ on CL–terms such that

$$\frac{}{\mathsf{I}\,t \to t} \qquad \frac{}{\mathsf{K}\,t\,u \to t} \qquad \frac{}{\mathsf{S}\,t\,u\,v \to t\,v\,(u\,v)} \qquad \frac{t \to u}{t\,v \to u\,v} \qquad \frac{t \to u}{v\,t \to v\,u}$$

for all CL–terms t, u, v

Definitions

- normal form is CL–term t such that $t \to u$ for no CL–term u
- $\to^+$ is transitive closure of $\to$
- $\to^*$ is transitive and reflexive closure of $\to$
- $t \to^! u$ if $t \to^* u$ for normal form u
- CL–term t is normalizing if $t \to^! u$ for some CL–term u
- infinite reduction is sequence $(t_i)_{i \geqslant 0}$ such that $t_i \to t_{i+1}$ for all $i \geqslant 0$
- CL–term t is terminating if there are no infinite reductions starting at t

Example

combinator $\mathsf{SII}(\mathsf{SII})$ is not terminating:

$$\mathsf{SII}(\mathsf{SII}) \to \mathsf{I}(\mathsf{SII})(\mathsf{I}(\mathsf{SII})) \to \mathsf{SII}(\mathsf{I}(\mathsf{SII})) \to \mathsf{SII}(\mathsf{SII})$$

Outline

Definition

$$\mathsf{B} = \mathsf{S}(\mathsf{KS})\mathsf{K} \qquad \mathsf{C} = \mathsf{S}(\mathsf{BBS})(\mathsf{KK}) \qquad \mathsf{Y} = \mathsf{B}(\mathsf{SI})(\mathsf{SII})(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))$$

Lemma

$$\mathsf{B}xyz \to^+ x(yz) \qquad \mathsf{C}xyz \to^+ xzy \qquad \mathsf{Y}x \to^+ x(\mathsf{Y}x)$$

Proof

$$\mathsf{B}xyz \to \mathsf{KS}x(\mathsf{K}x)yz \to \mathsf{S}(\mathsf{K}x)yz \to \mathsf{K}xz(yz) \to x(yz)$$

$$\mathsf{C}xyz \to \mathsf{BBS}x(\mathsf{KK}x)yz \to \mathsf{BBS}x\mathsf{K}yz \to^+ \mathsf{B}(\mathsf{S}x)\mathsf{K}yz \to^+ \mathsf{S}x(\mathsf{K}y)z \to xz(\mathsf{K}yz) \to xzy$$

$$\begin{aligned}\mathsf{Y}x &\to^+ \mathsf{SI}(\mathsf{SII}(\mathsf{B}(\mathsf{SI})(\mathsf{SII})))x \to \mathsf{I}x(\mathsf{SII}(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))x) \to^+ x(\mathsf{I}(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))(\mathsf{I}(\mathsf{B}(\mathsf{SI})(\mathsf{SII})))x) \\ &\to^+ x(\mathsf{B}(\mathsf{SI})(\mathsf{SII})(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))x) = x(\mathsf{Y}x)\end{aligned}$$

Outline

Definition

for all $n \geqslant 0$ Church numeral $\underline{n}$ is combinator

$$\underline{n} = \begin{cases} \mathsf{KI} & \text{if } n = 0 \\ \mathsf{SB}\,\underline{n-1} & \text{if } n > 0 \end{cases} \qquad \underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Definition

function $f\colon \mathbb{N}^n \to \mathbb{N}$ is CL–representable if there exists combinator F such that

$$f(x_1,\dots,x_n) = y \implies F\,\underline{x_1}\cdots\underline{x_n} \to^* \underline{y}$$

$$f(x_1,\dots,x_n) \text{ is undefined} \implies F\,\underline{x_1}\cdots\underline{x_n} \text{ is not normalizing}$$

for all $x_1,\dots,x_n,y \in \mathbb{N}$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

$\underline{n}\,x\,y \to^+ x^n\,y$

Proof

induction on $n \geqslant 0$

$$\underline{0}\,x\,y = \mathsf{KI}\,x\,y \to \mathsf{I}\,y \to y = x^0\,y$$

$$\underline{n+1}\,x\,y = \mathsf{SB}\,\underline{n}\,x\,y \to \mathsf{B}\,x\,(\underline{n}\,x)\,y \to^+ x\,(\underline{n}\,x\,y) \to^+ x\,(x^n\,y) = x^{n+1}\,y$$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

initial functions are CL–representable

Proof

$$\mathsf{zero} = \mathsf{K}(\mathsf{KI}): \qquad \mathsf{zero}\;\underline{x} \to \mathsf{KI} = \underline{0}$$

$$\mathsf{succ} = \mathsf{SB}: \qquad \mathsf{succ}\;\underline{x} = \mathsf{SB}\,\underline{x} = \underline{x+1}$$

$$\pi_i^n = \mathsf{K}^{i-1}((\mathsf{BK})^{n-i}\,\mathsf{I})$$

Outline

Definition (Bracket Abstraction)

- CL–term $[x]t$ is defined for all CL–terms t and variables x:

$$[x]t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}t & \text{if } x \notin \mathcal{V}\mathrm{ar}(t) \\ \mathsf{S}([x]t_1)([x]t_2) & \text{if } t = t_1 t_2 \text{ and } x \in \mathcal{V}\mathrm{ar}(t) \end{cases}$$

- $[x_1 \dots x_n]t$ abbreviates $[x_1](\dots [x_n]t \dots)$

Example

$$\begin{aligned} [x\,y\,z](xzy) &= [x\,y](\mathsf{S}([z](xz))([z]y)) = [x\,y](\mathsf{S}(\mathsf{S}([z]x)([z]z))(\mathsf{K}y)) = [x\,y](\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})(\mathsf{K}y)) \\ &= [x](\mathsf{S}([y](\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})))([y](\mathsf{K}y))) = [x](\mathsf{S}(\mathsf{K}(\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})))(\mathsf{S}([y]\mathsf{K})([y]y))) \\ &= [x](\mathsf{S}(\mathsf{K}(\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})))(\mathsf{S}(\mathsf{K}\mathsf{K})\mathsf{I})) = \cdots \\ &= \mathsf{S}(\mathsf{S}(\mathsf{K}\mathsf{S})(\mathsf{S}(\mathsf{K}\mathsf{K})(\mathsf{S}(\mathsf{K}\mathsf{S})(\mathsf{S}(\mathsf{S}(\mathsf{K}\mathsf{S})(\mathsf{S}(\mathsf{K}\mathsf{K})\mathsf{I}))(\mathsf{K}\mathsf{I})))))(\mathsf{K}(\mathsf{S}(\mathsf{K}\mathsf{K})\mathsf{I})) \end{aligned}$$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$

Lemma

$([x]t)\,x \to^* t$ for all CL–terms t and variables x

Proof

induction on t

- $t = x \implies ([x]t)\,x = \mathsf{I}x \to x = t$
- $x \notin \mathcal{V}\mathrm{ar}(t) \implies ([x]t)\,x = \mathsf{K}tx \to t$
- $t = t_1 t_2$ and $x \in \mathcal{V}\mathrm{ar}(t) \implies ([x]t_1)x \to^* t_1$ and $([x]t_2)x \to^* t_2$ by induction hypothesis
  $\implies ([x]t)\,x = \mathsf{S}([x]t_1)([x]t_2)x \to ([x]t_1)x(([x]t_2)x) \to^* t_1 t_2 = t$

Corollary (Combinatorial Completeness)

for every CL–term t with $\mathcal{V}\mathrm{ar}(t) = \{x_1, \dots, x_n\}$

1. $\exists$ combinator C such that $C\,x_1 \cdots x_n \to^* t$
2. $\exists$ combinator D such that $D\,x_2 \cdots x_n \to^* t[D/x_1]$

Proof

① $C = [x_1 \dots x_n]t$: $\quad C\,x_1 \cdots x_n \to^+ ([x_2 \dots x_n]t)\,x_2 \cdots x_n \to^+ \cdots \to^+ t$

② $D = \mathsf{Y}C$: $\quad D\,x_2 \cdots x_n \to^+ C\,D\,x_2 \cdots x_n \to^+ t[D/x_1]$

Definition (Bracket Abstraction, Optimized)

- CL–term $\langle x \rangle t$ is defined for all CL–terms t and variables x:

$$\langle x \rangle t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}t & \text{if } x \notin \mathcal{V}\text{ar}(t) \end{cases} \qquad \langle x \rangle t = \begin{cases} u & \text{if } t = u\,x \text{ and } x \notin \mathcal{V}\text{ar}(u) \\ \mathsf{B}u(\langle x \rangle v) & \text{if } t = u\,v \text{ and } x \notin \mathcal{V}\text{ar}(u) \\ \mathsf{C}(\langle x \rangle u)v & \text{if } t = u\,v \text{ and } x \notin \mathcal{V}\text{ar}(v) \\ \mathsf{S}(\langle x \rangle u)(\langle x \rangle v) & \text{if } t = u\,v \text{ and } x \in \mathcal{V}\text{ar}(u) \cap \mathcal{V}\text{ar}(v) \end{cases}$$

- $\langle x_1 \dots x_n \rangle t$ abbreviates $\langle x_1 \rangle (\dots \langle x_n \rangle t \dots)$

Example

$$\langle x\,y\,z \rangle (xzy) = \langle x\,y \rangle (\mathsf{C}(\langle z \rangle (xz))y) = \langle x\,y \rangle (\mathsf{C}xy) = \langle x \rangle (\mathsf{C}x) = \mathsf{C}$$

Lemma

$(\langle x \rangle t)\,x \to^* t$ for all CL–terms t and variables x

Outline

Lemma

CL–representable functions are closed under composition

Proof

$$f(x_1,\dots,x_n) = g(h_1(x_1,\dots,x_n),\dots,h_m(x_1,\dots,x_n))$$

with $G, H_1, \dots, H_m$ representing $g, h_1, \dots, h_m$

$$F = \langle x_1 \dots x_n \rangle (G\,(H_1\,x_1 \cdots x_n) \cdots (H_m\,x_1 \cdots x_n))$$

Lemma

CL–representable functions are closed under primitive recursion

$$f(0, y_1,\dots,y_n) = g(y_1,\dots,y_n)$$
$$f(x+1, y_1,\dots,y_n) = h(f(x,y_1,\dots,y_n), x, y_1,\dots,y_n)$$

$$f(x, y_1,\dots,y_n) = \begin{cases} g(y_1,\dots,y_n) & \text{if } x = 0 \\ h(f(x-1,y_1,\dots,y_n), x-1, y_1,\dots,y_n) & \text{otherwise} \end{cases}$$

Booleans

$\mathsf{T} = \mathsf{K}$ $\qquad \mathsf{F} = \mathsf{KI}$ $\qquad \mathsf{T}xy \to x$ $\qquad \mathsf{F}xy \to \mathsf{I}y \to y$

Test for Zero

$$\mathsf{zero?} = \mathsf{C}(\mathsf{B}(\mathsf{CIK}))(\mathsf{K}(\mathsf{KI}))$$

$$\mathsf{zero?}\ \underline{0} \to^* \mathsf{B}(\mathsf{CIK})\,\underline{0}\,(\mathsf{K}(\mathsf{KI})) \to^* \mathsf{CIK}(\underline{0}\,(\mathsf{K}(\mathsf{KI}))) \to \mathsf{CIKI} \to^* \mathsf{IIK} \to^* \mathsf{K} = \mathsf{T}$$

$$\begin{aligned} \mathsf{zero?}\ \underline{n+1} &\to^* \mathsf{B}(\mathsf{CIK})\,\underline{n+1}\,(\mathsf{K}(\mathsf{KI})) \to^* \mathsf{CIK}(\mathsf{SB}\,\underline{n}\,(\mathsf{K}(\mathsf{KI}))) \to^* \mathsf{CIK}(\mathsf{B}(\mathsf{K}(\mathsf{KI}))(\underline{n}\,(\mathsf{K}(\mathsf{KI})))) \\ &\to^* \mathsf{I}(\mathsf{B}(\mathsf{K}(\mathsf{KI}))(\underline{n}\,(\mathsf{K}(\mathsf{KI}))))\mathsf{K} \to \mathsf{B}(\mathsf{K}(\mathsf{KI}))(\underline{n}\,(\mathsf{K}(\mathsf{KI})))\mathsf{K} \\ &\to^* \mathsf{K}(\mathsf{KI})(\underline{n}\,(\mathsf{K}(\mathsf{KI}))\mathsf{K}) \to \mathsf{KI} = \mathsf{F} \end{aligned}$$

$$f(x, y_1, \ldots, y_n) = \begin{cases} g(y_1, \ldots, y_n) & \text{if } x = 0 \\ h(f(x-1, y_1, \ldots, y_n), x-1, y_1, \ldots, y_n) & \text{otherwise} \end{cases}$$

Predecessor

$$\mathsf{P} = \mathsf{C(C(CI(B(CI)(CI(SB))))(K(KI)))I}$$

Lemma

P represents predecessor function

Lemma

CL–representable functions are closed under primitive recursion

Proof

$$f(0, y_1, \ldots, y_n) = g(y_1, \ldots, y_n)$$
$$f(x+1, y_1, \ldots, y_n) = h(f(x, y_1, \ldots, y_n), x, y_1, \ldots, y_n)$$

with G, H representing g, h

$$F\,x\,y_1 \cdots y_n = (\mathsf{zero?}\,x)\,(G\,y_1 \cdots y_n)\,(H\,(F\,(\mathsf{P}\,x)\,y_1 \cdots y_n)\,(\mathsf{P}\,x)\,y_1 \cdots y_n)$$
$$F = \langle x\,y_1 \ldots y_n \rangle ((\mathsf{zero?}\,x)\,(G\,y_1 \cdots y_n)\,(H\,(F\,(\mathsf{P}\,x)\,y_1 \cdots y_n)\,(\mathsf{P}\,x)\,y_1 \cdots y_n))$$
$$F = \mathsf{Y}\,(\langle f\,x\,y_1 \ldots y_n \rangle ((\mathsf{zero?}\,x)\,(G\,y_1 \cdots y_n)\,(H\,(f\,(\mathsf{P}\,x)\,y_1 \cdots y_n)\,(\mathsf{P}\,x)\,y_1 \cdots y_n)))$$

Corollary

primitive recursive functions are CL–representable

Outline

Definition

function $f\colon \mathbb{N}^n \to \mathbb{N}$ is CL–representable if there exists combinator F such that

$$f(x_1, \ldots, x_n) = y \implies F\,\underline{x_1} \cdots \underline{x_n} \to^* \underline{y}$$
$$f(x_1, \ldots, x_n) \text{ is undefined} \implies F\,\underline{x_1} \cdots \underline{x_n} \text{ is not normalizing}$$

for all $x_1, \ldots, x_n, y \in \mathbb{N}$

Question

is it possible that both $F\,\underline{x_1} \cdots \underline{x_n} \to^* \underline{y}$ and $F\,\underline{x_1} \cdots \underline{x_n} \to^* \underline{z}$ with $y \neq z$?

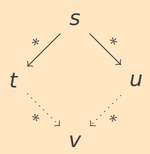
Answer

no, because CL has unique normal forms: $\forall$ term t $\forall$ normal forms n_1 and n_2

$$t \to^* n_1 \;\wedge\; t \to^* n_2 \implies n_1 = n_2$$

Theorem

CL is confluent: $\forall s\, \forall t\, \forall u\, [\, s \to^* t \land s \to^* u \implies \exists v\, (t \to^* v \land u \to^* v)\,]$



Corollary

CL has unique normal forms

Outline

Important Concepts

- $\to$
- $\to^+$
- $\to^*$
- $\to^!$
- B
- $[x]t$
- $\langle x \rangle t$
- bracket abstraction
- C
- Church numeral
- CL–representable function
- CL–term
- combinator
- combinatorial completeness
- combinatory logic
- confluence
- I
- F
- K
- normal form
- normalization
- P
- π_i^n
- S
- succ
- T
- termination
- unique normal forms
- (weak) reduction
- Y
- zero
- zero?

homework for November 20