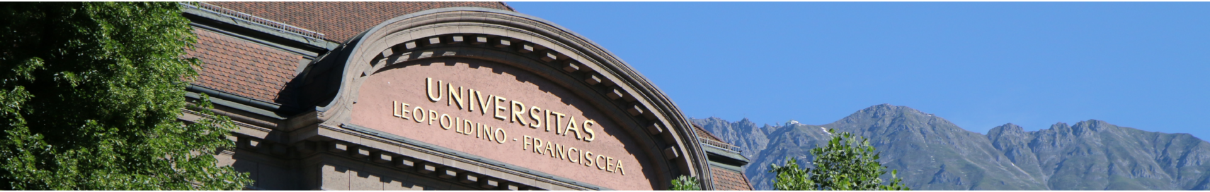
universität innsbruck

WS 2023 lecture 8



Computability Theory

Aart Middeldorp

Outline

Definitions (Combinatory Logic)

- CL–terms are built from
  - infinite set of variables $\mathcal{V} = \{x, y, z, \dots\}$
  - constants $\mathsf{I} \quad \mathsf{K} \quad \mathsf{S}$
  - application $s\,t$ for CL–terms s and t
- combinator is CL–term t without variables
- (weak) reduction is smallest relation $\to$ on CL–terms such that

$$\frac{}{\mathsf{I}\,t \to t} \qquad \frac{}{\mathsf{K}\,t\,u \to t} \qquad \frac{}{\mathsf{S}\,t\,u\,v \to t\,v\,(u\,v)} \qquad \frac{t \to u}{t\,v \to u\,v} \qquad \frac{t \to u}{v\,t \to v\,u}$$

  for all CL–terms t, u, v
- normal form is CL–term t such that $t \to u$ for no CL–term u
- $\to^{+}$ is transitive closure of $\to$
- $\to^{*}$ is transitive and reflexive closure of $\to$

Definitions

- $t \to^! u$ if $t \to^* u$ for normal form u
- CL–term t is normalizing if $t \to^! u$ for some CL–term u
- infinite reduction is sequence $(t_i)_{i \geqslant 0}$ such that $t_i \to t_{i+1}$ for all $i \geqslant 0$
- CL–term t is terminating if there are no infinite reductions starting at t
- $\mathsf{B} = \mathsf{S}(\mathsf{KS})\mathsf{K}$ $\quad$ $\mathsf{C} = \mathsf{S}(\mathsf{BBS})(\mathsf{KK})$ $\quad$ $\mathsf{Y} = \mathsf{B}(\mathsf{SI})(\mathsf{SII})(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))$
- for all $n \geqslant 0$ Church numeral $\underline{n}$ is combinator $(\mathsf{SB})^n(\mathsf{KI})$
- function $f\colon \mathbb{N}^n \to \mathbb{N}$ is CL–representable if there exists combinator F such that

$$\begin{aligned} f(x_1,\dots,x_n) = y \quad &\Longrightarrow \quad F\,\underline{x_1}\cdots\underline{x_n} \to^* \underline{y} \\ f(x_1,\dots,x_n) \text{ is undefined} \quad &\Longrightarrow \quad F\,\underline{x_1}\cdots\underline{x_n} \text{ is not normalizing} \end{aligned}$$

for all $x_1,\dots,x_n,y \in \mathbb{N}$

Lemma

$$\mathsf{B}xyz \to^+ x(yz) \qquad \mathsf{C}xyz \to^+ xzy \qquad \mathsf{Y}x \to^+ x(\mathsf{Y}x)$$

Definition (Bracket Abstraction)

CL–term $[x]t$ is defined for all CL–terms t and variables x:

$$[x]t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}t & \text{if } x \notin \mathcal{V}\mathrm{ar}(t) \\ \mathsf{S}([x]t_1)([x]t_2) & \text{if } t = t_1t_2 \text{ and } x \in \mathcal{V}\mathrm{ar}(t) \end{cases}$$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$ and $([x]t)x \to^* t$ for all CL–terms t and variables x

Corollary (Combinatorial Completeness)

for every CL–term t with $\mathcal{V}\mathrm{ar}(t) = \{x_1, \dots, x_n\}$

1. $\exists$ combinator C such that $C\,x_1 \cdots x_n \to^* t$
2. $\exists$ combinator D such that $D\,x_2 \cdots x_n \to^* t[D/x_1]$

Definition (Bracket Abstraction, Optimized)

- CL–term $\langle x \rangle t$ is defined for all CL–terms t and variables x:

$$\langle x \rangle t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}t & \text{if } x \notin \mathcal{V}\mathrm{ar}(t) \\ u & \text{if } t = u\,x \text{ and } x \notin \mathcal{V}\mathrm{ar}(u) \\ \mathsf{B}u(\langle x \rangle v) & \text{if } t = u\,v \text{ and } x \notin \mathcal{V}\mathrm{ar}(u) \\ \mathsf{C}(\langle x \rangle u)\,v & \text{if } t = u\,v \text{ and } x \notin \mathcal{V}\mathrm{ar}(v) \\ \mathsf{S}(\langle x \rangle u)(\langle x \rangle v) & \text{if } t = u\,v \text{ and } x \in \mathcal{V}\mathrm{ar}(u) \cap \mathcal{V}\mathrm{ar}(v) \end{cases}$$

- $\langle x_1 \dots x_n \rangle t$ abbreviates $\langle x_1 \rangle (\dots \langle x_n \rangle t \dots)$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$ and $(\langle x \rangle t)\,x \to^* t$ for all CL–terms t and variables x

Definition

$$\mathsf{T} = \mathsf{K} \qquad \mathsf{F} = \mathsf{KI} \qquad \mathsf{zero?} = \mathsf{C(B(CIK))(K(KI))}$$

Lemmata

1. initial functions are CL–representable
2. CL–representable functions are closed under composition and primitive recursion

Theorem

CL is confluent: $\forall s\, \forall t\, \forall u\, \left[\, s \to^* t \,\wedge\, s \to^* u \implies \exists v\, (t \to^* v \,\wedge\, u \to^* v)\,\right]$

Corollary

CL has unique normal forms

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability,
while programs, ...

Part II: Combinatory Logic and Lambda Calculus

α–equivalence, abstraction, arithmetization, β–reduction, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, de Bruijn notation, η–reduction, fixed point theorem, intuitionistic propositional logic, λ–definability, normalization theorem, termination, typing, undecidability, Z property, ...

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Curry–Howard isomorphism, de Bruijn notation, η–reduction, fixed point theorem, intuitionistic propositional logic, λ–definability, normalization theorem, termination, typing, undecidability, Z property, . . .

Outline

Definition (Parallel Reduction)

$$\frac{}{t \twoheadrightarrow t}$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

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$$\frac{}{t \twoheadrightarrow\!\!\!\!\!\!\;\shortmid\!\shortmid\; t}$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

$$\frac{}{\mathsf{I}\,t \twoheadrightarrow\!\!\!\!\!\!\;\shortmid\!\shortmid\; t}$$

for all CL–terms t

Definition (Parallel Reduction)

$$\frac{}{t \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\;\; t}$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

$$\frac{}{\mathsf{I}\, t \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\;\; t} \qquad\qquad \frac{}{\mathsf{K}\, t\, u \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\;\; t}$$

for all CL–terms t, u

Definition (Parallel Reduction)

$$\frac{}{t \mathrel{+\!\!\!\!\!\shortmid\!\!\!\shortmid\!\!\rightarrow} t}$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

$$\frac{}{\mathsf{I}\, t \mathrel{+\!\!\!\!\!\shortmid\!\!\!\shortmid\!\!\rightarrow} t} \qquad \frac{}{\mathsf{K}\, t\, u \mathrel{+\!\!\!\!\!\shortmid\!\!\!\shortmid\!\!\rightarrow} t} \qquad \frac{}{\mathsf{S}\, t\, u\, v \mathrel{+\!\!\!\!\!\shortmid\!\!\!\shortmid\!\!\rightarrow} t\, v\, (u\, v)}$$

for all CL–terms t, u, v

Definition (Parallel Reduction)

$$\frac{}{t \;\twoheadrightarrow\!\!\!\!\!\!\;+\; t}$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

$$\frac{}{\mathsf{I}\,t \;\twoheadrightarrow\!\!\!\!\!\!\;+\; t} \qquad \frac{}{\mathsf{K}\,t\,u \;\twoheadrightarrow\!\!\!\!\!\!\;+\; t} \qquad \frac{}{\mathsf{S}\,t\,u\,v \;\twoheadrightarrow\!\!\!\!\!\!\;+\; t\,v\,(u\,v)}$$

for all CL–terms t, u, v

$$\frac{t_1 \;\twoheadrightarrow\!\!\!\!\!\!\;+\; u_1 \qquad t_2 \;\twoheadrightarrow\!\!\!\!\!\!\;+\; u_2}{t_1\,t_2 \;\twoheadrightarrow\!\!\!\!\!\!\;+\; u_1\,u_2}$$

for all CL–terms t_1, t_2, u_1, u_2

Definition (Parallel Reduction)

$$\frac{}{t \;\twoheadrightarrow\!\!\!\!\!\!+\; t}$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

$$\frac{}{\mathsf{I}\,t \;\twoheadrightarrow\!\!\!\!\!\!+\; t} \qquad \frac{}{\mathsf{K}\,t\,u \;\twoheadrightarrow\!\!\!\!\!\!+\; t} \qquad \frac{}{\mathsf{S}\,t\,u\,v \;\twoheadrightarrow\!\!\!\!\!\!+\; t\,v\,(u\,v)}$$

for all CL–terms t, u, v

$$\frac{t_1 \;\twoheadrightarrow\!\!\!\!\!\!+\; u_1 \qquad t_2 \;\twoheadrightarrow\!\!\!\!\!\!+\; u_2}{t_1\,t_2 \;\twoheadrightarrow\!\!\!\!\!\!+\; u_1\,u_2}$$

for all CL–terms t_1, t_2, u_1, u_2

Lemma

$\rightarrow \;\subseteq\; \twoheadrightarrow\!\!\!\!\!\!+ \;\subseteq\; \rightarrow^*$

Example

$\mathsf{K}(\mathsf{IK})(\mathsf{IK}(\mathsf{KSI})) \not\twoheadrightarrow \mathsf{KK}(\mathsf{KS})$

Example

$$\mathsf{K(IK)(IK(KSI))} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{KK(KS)}: \qquad \frac{\dfrac{\overline{\mathsf{K} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{K}} \qquad \overline{\mathsf{IK} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{K}}}{\mathsf{K(IK)} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{KK}} \qquad \dfrac{\overline{\mathsf{IK} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{K}} \qquad \overline{\mathsf{KSI} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{S}}}{\mathsf{IK(KSI)} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{KS}}}{\mathsf{K(IK)(IK(KSI))} \mathrel{-\!\!\!\parallel\!\!\!\rightarrow} \mathsf{KK(KS)}}$$

Example

$$\mathsf{K(IK)(IK(KSI))} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{KK(KS)}: \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{K} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{K}} \quad \dfrac{}{\mathsf{IK} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{K}}}{\mathsf{K(IK)} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{KK}} \qquad \dfrac{\dfrac{}{\mathsf{IK} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{K}} \quad \dfrac{}{\mathsf{KSI} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{S}}}{\mathsf{IK(KSI)} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{KS}}}{\mathsf{K(IK)(IK(KSI))} \mathrel{\parallel\!\!\!\rightarrow} \mathsf{KK(KS)}}$$

Lemma

parallel reduction has diamond property: $\forall$ terms s, t, u $\exists$ term v

$$\begin{tikzcd} & s \arrow[dl, "\parallel"'] \arrow[dr, "\parallel"] & \\ t \arrow[dr, dotted, "\parallel"'] & & u \arrow[dl, dotted, "\parallel"] \\ & v & \end{tikzcd}$$

Definition (Parallel Reduction)

$$\frac{}{t \mathrel{\twoheadrightarrow\!\!\!\!\!\!+} t} \qquad \langle 1 \rangle$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

$$\frac{}{\mathsf{I}\, t \mathrel{+\!\!\!\!\twoheadrightarrow} t} \qquad \frac{}{\mathsf{K}\, t\, u \mathrel{+\!\!\!\!\twoheadrightarrow} t} \qquad \frac{}{\mathsf{S}\, t\, u\, v \mathrel{+\!\!\!\!\twoheadrightarrow} t\, v\, (u\, v)} \qquad \langle 2 \rangle$$

for all CL–terms t, u, v

$$\frac{t_1 \mathrel{+\!\!\!\!\twoheadrightarrow} u_1 \qquad t_2 \mathrel{+\!\!\!\!\twoheadrightarrow} u_2}{t_1\, t_2 \mathrel{+\!\!\!\!\twoheadrightarrow} u_1\, u_2} \qquad \langle 3 \rangle$$

for all CL–terms t_1, t_2, u_1, u_2

Lemma

$\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\twoheadrightarrow} \;\subseteq\; \rightarrow^*$

Lemma

$$\forall\, s\, \forall\, t\, \forall\, u\, \left[\, s \twoheadrightarrow\!\!\!\!\!\!\!\!\parallel\; t \,\wedge\, s \twoheadrightarrow\!\!\!\!\!\!\!\!\parallel\; u \quad\Longrightarrow\quad \exists\, v\, (t \twoheadrightarrow\!\!\!\!\!\!\!\!\parallel\; v \,\wedge\, u \twoheadrightarrow\!\!\!\!\!\!\!\!\parallel\; v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!\!\!\parallel\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!\!\!\parallel\; u$

Lemma

$$\forall\, s\, \forall\, t\, \forall\, u \left[\, s \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+} t \;\land\; s \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+} u \quad\Longrightarrow\quad \exists\, v \,(t \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+} v \;\land\; u \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+} v)\,\right]$$

Proof

- induction on derivation of $s \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+} t$ and $s \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+} u$
- easy cases: $s \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+}^{\langle 1 \rangle} t$ or $s \mathrel{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!+}^{\langle 1 \rangle} u$

Lemma

$\forall\, s\, \forall\, t\, \forall\, u\, \left[\, s \twoheadrightarrow\!\!\!\!+ \; t \,\wedge\, s \twoheadrightarrow\!\!\!\!+ \; u \quad\Longrightarrow\quad \exists\, v\, (t \twoheadrightarrow\!\!\!\!+ \; v \,\wedge\, u \twoheadrightarrow\!\!\!\!+ \; v)\,\right]$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!+ \; t$ and $s \twoheadrightarrow\!\!\!\!+ \; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} u$

Lemma

$$\forall s \, \forall t \, \forall u \left[s \twoheadrightarrow\!\!\!\!\!\!+\, t \wedge s \twoheadrightarrow\!\!\!\!\!\!+\, u \quad \Longrightarrow \quad \exists v \, (t \twoheadrightarrow\!\!\!\!\!\!+\, v \wedge u \twoheadrightarrow\!\!\!\!\!\!+\, v) \right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!+\, t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\, u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1 \rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1 \rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2 \rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2 \rangle} u$
  or both $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 3 \rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 3 \rangle} u$

Lemma

$$\forall s\, \forall t\, \forall u\, \left[\, s \twoheadrightarrow\!\!\!\!\!\!+\; t \wedge s \twoheadrightarrow\!\!\!\!\!\!+\; u \quad\Longrightarrow\quad \exists v\, (t \twoheadrightarrow\!\!\!\!\!\!+\; v \wedge u \twoheadrightarrow\!\!\!\!\!\!+\; v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!+\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 1\rangle}\; t$ or $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 1\rangle}\; u$ or both $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 2\rangle}\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 2\rangle}\; u$
  or both $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 3\rangle}\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 3\rangle}\; u$
- interesting case (modulo symmetry): $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 2\rangle}\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\;^{\langle 3\rangle}\; u$

Lemma

$$\forall s\, \forall t\, \forall u\, \left[\, s \twoheadrightarrow\!\!\!\!\!\!\;+\; t \wedge s \,\#\!\!\!\rightarrow u \quad \Longrightarrow \quad \exists v\, (t \,\#\!\!\!\rightarrow v \wedge u \,\#\!\!\!\rightarrow v)\,\right]$$

Proof

- induction on derivation of $s \,\#\!\!\!\rightarrow t$ and $s \,\#\!\!\!\rightarrow u$
- easy cases: $s \,\#\!\!\!\rightarrow^{\langle 1 \rangle} t$ or $s \,\#\!\!\!\rightarrow^{\langle 1 \rangle} u$ or both $s \,\#\!\!\!\rightarrow^{\langle 2 \rangle} t$ and $s \,\#\!\!\!\rightarrow^{\langle 2 \rangle} u$
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  $s = s_1 s_2$ and $u = u_1 u_2$ with $s_1 \,\#\!\!\!\rightarrow u_1$ and $s_2 \,\#\!\!\!\rightarrow u_2$

Lemma

$$\forall s\,\forall t\,\forall u\,\left[\,s \twoheadrightarrow\!\!\!\!+ \; t \wedge s \twoheadrightarrow\!\!\!\!+ \; u \quad\Longrightarrow\quad \exists v\,(t \twoheadrightarrow\!\!\!\!+ \; v \wedge u \twoheadrightarrow\!\!\!\!+ \; v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!+ \; t$ and $s \twoheadrightarrow\!\!\!\!+ \; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} u$
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- interesting case (modulo symmetry): $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!+^{\langle 3\rangle} u$

  $s = s_1 s_2$ and $u = u_1 u_2$ with $s_1 \twoheadrightarrow\!\!\!\!+ \; u_1$ and $s_2 \twoheadrightarrow\!\!\!\!+ \; u_2$

  ① $s_1 = \mathsf{I}$

  ② $s_1 = \mathsf{K}\,s'$

  ③ $s_1 = \mathsf{S}\,s'\,s''$

Lemma

$$\forall s\, \forall t\, \forall u\, \left[\, s \twoheadrightarrow\!\!\!\!\!\!+\; t \,\wedge\, s \twoheadrightarrow\!\!\!\!\!\!+\; u \quad\Longrightarrow\quad \exists v\, (t \twoheadrightarrow\!\!\!\!\!\!+\; v \,\wedge\, u \twoheadrightarrow\!\!\!\!\!\!+\; v) \,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!+\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1 \rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1 \rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2 \rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2 \rangle} u$
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- interesting case (modulo symmetry): $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2 \rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 3 \rangle} u$

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  ① $s_1 = \mathsf{I}$ $\qquad t = s_2$

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Lemma

$$\forall s \, \forall t \, \forall u \, \big[\, s \twoheadrightarrow\!\!\!\!\!\!\shortmid\ t \, \wedge \, s \twoheadrightarrow\!\!\!\!\!\!\shortmid\ u \quad \Longrightarrow \quad \exists v \, (t \twoheadrightarrow\!\!\!\!\!\!\shortmid\ v \, \wedge \, u \twoheadrightarrow\!\!\!\!\!\!\shortmid\ v) \,\big]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!\shortmid\ t$ and $s \twoheadrightarrow\!\!\!\!\!\!\shortmid\ u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!\shortmid^{\langle 1 \rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!\!\shortmid^{\langle 1 \rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!\!\shortmid^{\langle 2 \rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!\shortmid^{\langle 2 \rangle} u$
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  ① $s_1 = \mathsf{I} \qquad t = s_2 \qquad u_1 = \mathsf{I}$

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Lemma

$$\forall s \, \forall t \, \forall u \left[s \nrightarrow\!\!\!\!\rightarrow t \wedge s \nrightarrow\!\!\!\!\rightarrow u \quad \Longrightarrow \quad \exists v \, (t \nrightarrow\!\!\!\!\rightarrow v \wedge u \nrightarrow\!\!\!\!\rightarrow v) \right]$$

Proof

- induction on derivation of $s \nrightarrow\!\!\!\!\rightarrow t$ and $s \nrightarrow\!\!\!\!\rightarrow u$
- easy cases: $s \nrightarrow\!\!\!\!\rightarrow^{\langle 1 \rangle} t$ or $s \nrightarrow\!\!\!\!\rightarrow^{\langle 1 \rangle} u$ or both $s \nrightarrow\!\!\!\!\rightarrow^{\langle 2 \rangle} t$ and $s \nrightarrow\!\!\!\!\rightarrow^{\langle 2 \rangle} u$
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- interesting case (modulo symmetry): $s \nrightarrow\!\!\!\!\rightarrow^{\langle 2 \rangle} t$ and $s \nrightarrow\!\!\!\!\rightarrow^{\langle 3 \rangle} u$

  $s = s_1 s_2$ and $u = u_1 u_2$ with $s_1 \nrightarrow\!\!\!\!\rightarrow u_1$ and $s_2 \nrightarrow\!\!\!\!\rightarrow u_2$

  ① $s_1 = \mathsf{I}$ $\quad t = s_2$ $\quad u_1 = \mathsf{I}$ $\quad u \nrightarrow\!\!\!\!\rightarrow u_2$

  ② $s_1 = \mathsf{K}\, s'$

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Lemma

$$\forall s\, \forall t\, \forall u\, \big[\, s \twoheadrightarrow\!\!\!\!\!\!+\; t \;\wedge\; s \twoheadrightarrow\!\!\!\!\!\!+\; u \quad\Longrightarrow\quad \exists v\, (t \twoheadrightarrow\!\!\!\!\!\!+\; v \;\wedge\; u \twoheadrightarrow\!\!\!\!\!\!+\; v)\,\big]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!+\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2\rangle} u$
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Lemma

$$\forall s\, \forall t\, \forall u\, \left[\, s \twoheadrightarrow\!\!\!\!\!+\; t \wedge s \twoheadrightarrow\!\!\!\!\!+\; u \quad\Longrightarrow\quad \exists v\, (t \twoheadrightarrow\!\!\!\!\!+\; v \wedge u \twoheadrightarrow\!\!\!\!\!+\; v)\, \right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!+\; t$ and $s \twoheadrightarrow\!\!\!\!\!+\; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} u$
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  $s = s_1 s_2$ and $u = u_1 u_2$ with $s_1 \twoheadrightarrow\!\!\!\!\!+\; u_1$ and $s_2 \twoheadrightarrow\!\!\!\!\!+\; u_2$

  ① $s_1 = \mathsf{I}$ $\quad t = s_2$ $\quad u_1 = \mathsf{I}$ $\quad u \twoheadrightarrow\!\!\!\!\!+\; u_2$ $\quad t \twoheadrightarrow\!\!\!\!\!+\; u_2$

  ② $s_1 = \mathsf{K}\, s'$ $\quad t = s'$

  ③ $s_1 = \mathsf{S}\, s'\, s''$

Lemma

$$\forall s\,\forall t\,\forall u\,\left[\,s \twoheadrightarrow\!\!\!\!\!+\; t \;\wedge\; s \twoheadrightarrow\!\!\!\!\!+\; u \quad\Longrightarrow\quad \exists v\,(t \twoheadrightarrow\!\!\!\!\!+\; v \;\wedge\; u \twoheadrightarrow\!\!\!\!\!+\; v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!+\; t$ and $s \twoheadrightarrow\!\!\!\!\!+\; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} u$
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- interesting case (modulo symmetry): $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!+^{\langle 3\rangle} u$

  $s = s_1 s_2$ and $u = u_1 u_2$ with $s_1 \twoheadrightarrow\!\!\!\!\!+\; u_1$ and $s_2 \twoheadrightarrow\!\!\!\!\!+\; u_2$

  ① $s_1 = \mathsf{I}$ $\quad t = s_2$ $\quad u_1 = \mathsf{I}$ $\quad u \twoheadrightarrow\!\!\!\!\!+\; u_2$ $\quad t \twoheadrightarrow\!\!\!\!\!+\; u_2$

  ② $s_1 = \mathsf{K}\,s'$ $\quad t = s'$ $\quad u_1 = \mathsf{K}\,u'$ with $s' \twoheadrightarrow\!\!\!\!\!+\; u'$

  ③ $s_1 = \mathsf{S}\,s'\,s''$

Lemma

$$\forall s\,\forall t\,\forall u\,\big[\,s \twoheadrightarrow\!\!\!\!\!+ \ t \wedge s \twoheadrightarrow\!\!\!\!\!+ \ u \quad\Longrightarrow\quad \exists v\,(t \twoheadrightarrow\!\!\!\!\!+ \ v \wedge u \twoheadrightarrow\!\!\!\!\!+ \ v)\,\big]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!+ \ t$ and $s \twoheadrightarrow\!\!\!\!\!+ \ u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} u$
  or both $s \twoheadrightarrow\!\!\!\!\!+^{\langle 3\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!+^{\langle 3\rangle} u$
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  ① $s_1 = \mathsf{I}$ $\quad t = s_2$ $\quad u_1 = \mathsf{I}$ $\quad u \twoheadrightarrow\!\!\!\!\!+ \ u_2$ $\quad t \twoheadrightarrow\!\!\!\!\!+ \ u_2$

  ② $s_1 = \mathsf{K}\,s'$ $\quad t = s'$ $\quad u_1 = \mathsf{K}\,u'$ with $s' \twoheadrightarrow\!\!\!\!\!+ \ u'$ $\quad u \twoheadrightarrow\!\!\!\!\!+ \ u'$

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Lemma

$$\forall s\, \forall t\, \forall u\, \left[s \twoheadrightarrow\!\!\!\!\!\!+\ t \wedge s \twoheadrightarrow\!\!\!\!\!\!+\ u \implies \exists v\, (t \twoheadrightarrow\!\!\!\!\!\!+\ v \wedge u \twoheadrightarrow\!\!\!\!\!\!+\ v) \right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!+\ t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\ u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1 \rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 1 \rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2 \rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!+^{\langle 2 \rangle} u$
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  ① $s_1 = \mathsf{I}$ $\quad t = s_2$ $\quad u_1 = \mathsf{I}$ $\quad u \twoheadrightarrow\!\!\!\!\!\!+\ u_2$ $\quad t \twoheadrightarrow\!\!\!\!\!\!+\ u_2$

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Lemma

$$\forall s\,\forall t\,\forall u\,\left[\,s \twoheadrightarrow\!\!\!\!+\, t \;\wedge\; s \twoheadrightarrow\!\!\!\!+\, u \quad\Longrightarrow\quad \exists\, v\,(t \twoheadrightarrow\!\!\!\!+\, v \;\wedge\; u \twoheadrightarrow\!\!\!\!+\, v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!+\, t$ and $s \twoheadrightarrow\!\!\!\!+\, u$
- easy cases: $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle}\, t$ or $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle}\, u$ or both $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle}\, t$ and $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle}\, u$
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  ③ $s_1 = \mathsf{S}\,s'\,s''$ $\quad t = s' s_2\,(s'' s_2)$

Lemma

$$\forall s\, \forall t\, \forall u\, \left[s \twoheadrightarrow\!\!\!\!\!\!+\;\; t \;\wedge\; s \twoheadrightarrow\!\!\!\!\!\!+\;\; u \quad\Longrightarrow\quad \exists v\, (t \twoheadrightarrow\!\!\!\!\!\!+\;\; v \;\wedge\; u \twoheadrightarrow\!\!\!\!\!\!+\;\; v) \right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!+\;\; t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\;\; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!+\;\;^{\langle 1\rangle}\, t$ or $s \twoheadrightarrow\!\!\!\!\!\!+\;\;^{\langle 1\rangle}\, u$ or both $s \twoheadrightarrow\!\!\!\!\!\!+\;\;^{\langle 2\rangle}\, t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\;\;^{\langle 2\rangle}\, u$
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  ③ $s_1 = \mathsf{S}\, s'\, s''$ $\quad t = s's_2\,(s''s_2)$ $\quad u_1 = \mathsf{S}\, u'\, u''$ with $s' \twoheadrightarrow\!\!\!\!\!\!+\;\; u'$ and $s'' \twoheadrightarrow\!\!\!\!\!\!+\;\; u''$

Lemma

$$\forall s\,\forall t\,\forall u\,\left[\, s \twoheadrightarrow\!\!\!\!\!+\; t \wedge s \twoheadrightarrow\!\!\!\!\!+\; u \quad\Longrightarrow\quad \exists v\,(t \twoheadrightarrow\!\!\!\!\!+\; v \wedge u \twoheadrightarrow\!\!\!\!\!+\; v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!+\; t$ and $s \twoheadrightarrow\!\!\!\!\!+\; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!+^{\langle 2\rangle} u$
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  $u \twoheadrightarrow\!\!\!\!\!+\; u'u_2\,(u''u_2)$

Lemma

$$\forall s\, \forall t\, \forall u\, \left[\, s \twoheadrightarrow\!\!\!\!\!\!\nparallel t \wedge s \twoheadrightarrow\!\!\!\!\!\!\nparallel u \quad\Longrightarrow\quad \exists\, v\, (t \twoheadrightarrow\!\!\!\!\!\!\nparallel v \wedge u \twoheadrightarrow\!\!\!\!\!\!\nparallel v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!\!\!\nparallel t$ and $s \twoheadrightarrow\!\!\!\!\!\!\nparallel u$
- easy cases: $s \twoheadrightarrow\!\!\!\!\!\!\nparallel^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!\!\!\nparallel^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!\!\!\nparallel^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!\!\!\nparallel^{\langle 2\rangle} u$
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  $u \twoheadrightarrow\!\!\!\!\!\!\nparallel u' u_2\, (u'' u_2)$ $\quad t \twoheadrightarrow\!\!\!\!\!\!\nparallel u' u_2\, (u'' u_2)$

Corollary

CL is confluent

Proof

$$\begin{tikzcd} \cdot \arrow[r, "*"] \arrow[d, "*"'] & \cdot \\ \cdot & \end{tikzcd}$$

Corollary

CL is confluent

Proof

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \\
\cdot \arrow[d] & & & & \\
\cdot \arrow[d] & & & & \\
\cdot & & & &
\end{tikzcd}
$$

Corollary

CL is confluent

Proof

$$\begin{tikzcd}
\cdot \arrow[r, "\shortparallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \\
\cdot \arrow[d, "=" marking] & & & & \\
\cdot \arrow[d, "=" marking] & & & & \\
\cdot & & & &
\end{tikzcd}$$

Corollary

CL is confluent

Proof

$$
\begin{tikzcd}
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \\
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot & & & \\
\cdot \arrow[d, "=" marking] & & & & \\
\cdot & & & &
\end{tikzcd}
$$

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Proof

$$
\begin{tikzcd}
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \\
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot & & \\
\cdot \arrow[d, "=" marking] & & & & \\
\cdot & & & &
\end{tikzcd}
$$

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$$\begin{tikzcd}
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \\
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot & \\
\cdot \arrow[d, "=" marking] & & & & \\
\cdot & & & &
\end{tikzcd}$$

Corollary

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Proof

$$
\begin{tikzcd}
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot
\end{tikzcd}
$$

Corollary

CL is confluent

Proof

$$
\begin{tikzcd}
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[d, "=" marking] & & & & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[d, "=" marking] & & & & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot
\end{tikzcd}
$$

Corollary

CL is confluent

Proof

$$
\begin{tikzcd}
\cdot \arrow[r, "\parallel" description] \arrow[d, "=" description] & \cdot \arrow[r, "\parallel" description] & \cdot \arrow[r, "\parallel" description] & \cdot \arrow[r, "\parallel" description] & \cdot \arrow[d, "*"] \\
\cdot \arrow[d, "=" description] & & & & \cdot \arrow[d, "*"] \\
\cdot \arrow[d, "=" description] & & & & \cdot \arrow[d, "*"] \\
\cdot \arrow[r, "*"'] & \cdot \arrow[r, "*"'] & \cdot \arrow[r, "*"'] & \cdot \arrow[r, "*"'] & \cdot
\end{tikzcd}
$$

Corollary

CL is confluent

Proof

$$
\begin{tikzcd}
\cdot \arrow[r, "\shortparallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \arrow[ddd, "*"] \\
\cdot \arrow[d, "=" marking] & & & & \\
\cdot \arrow[d, "=" marking] & & & & \\
\cdot \arrow[rrrr, "*"'] & & & & \cdot
\end{tikzcd}
$$

Definition (Conversion)

$\leftrightarrow^*$ is transitive, reflexive and symmetric closure of $\to$

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Church–Rosser Theorem

CL has Church–Rosser property

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Church–Rosser Theorem

CL has Church–Rosser property: $\forall\, t\, \forall\, u\, \left[\, t \leftrightarrow^* u \implies \exists\, v\, (t \to^* v \wedge u \to^* v)\,\right]$

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Church–Rosser Theorem

CL has Church–Rosser property: $\forall\, t\, \forall\, u\, \left[\, t \leftrightarrow^* u \implies \exists\, v\, (t \to^* v \wedge u \to^* v)\,\right]$

Proof

easy consequence of confluence

Outline

1. **Summary of Previous Lecture**
2. **Church–Rosser Theorem**
3. **Z Property**
4. **CL–Representability**
5. **Summary**

Definition

ARS $\mathcal{A} = \langle A, \to \rangle$ has **Z property** if

$$a \to b \quad\Longrightarrow\quad b \to^* \bullet(a) \to^* \bullet(b)$$

for some function $\bullet$ on A

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Notation

$a^\bullet$ for $\bullet(a)$

Definition

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$$\begin{tikzcd} a \arrow[r] & b \arrow[dl, dotted, "*"'] \\ a^\bullet \arrow[r, dotted, "*"] & b^\bullet \end{tikzcd}$$

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Notation

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Example

ARS

$$\begin{tikzcd} \mathsf{d} \arrow[rrrr, bend left] & \mathsf{a} \arrow[l] \arrow[r] & \mathsf{b} \arrow[r] & \mathsf{c} \arrow[r] & \mathsf{e} \arrow[loop right] \end{tikzcd}$$

Definition

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$$a \to b \quad \Longrightarrow \quad b \to^* \bullet(a) \to^* \bullet(b)$$

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\end{tikzcd}$$

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- define $\mathsf{a}^\bullet = \mathsf{b}^\bullet = \mathsf{c}^\bullet = \mathsf{d}^\bullet = \mathsf{e}^\bullet = \mathsf{e}$

Definition

ARS $\mathcal{A} = \langle A, \to \rangle$ has Z property if

$$a \to b \quad \Longrightarrow \quad b \to^* \bullet(a) \to^* \bullet(b)$$

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$$\begin{tikzcd}
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\mathsf{d} \arrow[rrrr, bend left] & \mathsf{a} \arrow[l] \arrow[r] & \mathsf{b} \arrow[r] & \mathsf{c} \arrow[r] & \mathsf{e} \arrow[loop right]
\end{tikzcd}$$

- define $\mathsf{a}^\bullet = \mathsf{b}^\bullet = \mathsf{c}^\bullet = \mathsf{d}^\bullet = \mathsf{e}^\bullet = \mathsf{e}$
- every element rewrites to e $\quad \Longrightarrow \quad$ Z property is trivially satisfied

Lemma (Monotonicity)

$a \to^* b \implies a^\bullet \to^* b^\bullet$ for every ARS $\langle A, \to \rangle$ with Z property for $\bullet$

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$a \to^* b \implies a^\bullet \to^* b^\bullet$ for every ARS $\langle A, \to \rangle$ with Z property for $\bullet$

Proof

induction on number of steps in $a \to^* b$

Lemma

every ARS with Z property is confluent

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Proof

$b \leftarrow a \rightarrow^n c \quad \Longrightarrow \quad b \downarrow c$ by induction on n:

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$b \leftarrow a \rightarrow^n c \implies b \downarrow c$ by induction on n:

- $n = 0 \implies c = a \rightarrow b$

Lemma

every ARS with Z property is confluent

Proof

$b \leftarrow a \rightarrow^n c \quad \Longrightarrow \quad b \downarrow c$ by induction on n:

- $n = 0 \quad \Longrightarrow \quad c = a \rightarrow b$
- $n > 0 \quad \Longrightarrow \quad a \rightarrow^{n-1} d \rightarrow c$ for some element d

Lemma

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$b \leftarrow a \rightarrow^n c \implies b \downarrow c$ by induction on n:

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$$\begin{tikzcd} a \arrow[rr, "n-1"] \arrow[d] & & d \arrow[r] & c \\ b & & & \end{tikzcd}$$

Lemma

every ARS with Z property is confluent

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$$
\begin{tikzcd}
a \arrow[rr, "n-1"] \arrow[d] \arrow[dr, dotted, no head, "\mathrm{Z}" description] & & d \arrow[r] & c \\
b \arrow[r, dashed, "*"'] & a^\bullet & &
\end{tikzcd}
$$

Lemma

every ARS with Z property is confluent

Proof

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- $n > 0 \quad \Longrightarrow \quad a \rightarrow^{n-1} d \rightarrow c$ for some element d

$$
\begin{tikzcd}
a \arrow[rr, "n-1"] \arrow[d] \arrow[dr, dotted, "\mathrm{Z}" description, no head] & & \color{red}{d} \arrow[r, red] \arrow[dr, dotted, "\mathrm{Z}" description, no head] & \color{red}{c} \arrow[d, dashed, "*"] \\
b \arrow[r, dashed, "*"'] & a^\bullet & & d^\bullet
\end{tikzcd}
$$

Lemma

every ARS with Z property is confluent

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$b \leftarrow a \to^n c \quad \Longrightarrow \quad b \downarrow c$ by induction on n:

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$$
\begin{tikzcd}
a \arrow[rr, "n-1"] \arrow[d] \arrow[dr, phantom, "\mathrm{Z}"] & & d \arrow[r] \arrow[dr, phantom, "\mathrm{Z}"] & c \arrow[d, dashed, "*"] \\
b \arrow[r, dashed, "*"'] & a^\bullet \arrow[rr, dashed, "*"'] & & d^\bullet
\end{tikzcd}
$$

Lemma

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$$
\begin{tikzcd}
a \arrow[rr, "n-1"] \arrow[d] \arrow[dr, dotted, no head, "\mathrm{Z}" description] & & d \arrow[r] \arrow[dr, dotted, no head, "\mathrm{Z}" description] & c \arrow[d, dashed, "*"] \\
b \arrow[r, dashed, "*"'] & a^\bullet \arrow[rr, dashed, "*"'] & & d^\bullet
\end{tikzcd}
$$

$\leftarrow \cdot \rightarrow^{*} \subseteq \downarrow$ (semi-confluence)

Lemma

every ARS with Z property is confluent

Proof

$b \leftarrow a \to^n c \implies b \downarrow c$ by induction on n:

- $n = 0 \implies c = a \to b$
- $n > 0 \implies a \to^{n-1} d \to c$ for some element d

$$\begin{tikzcd}
a \arrow[rr, "n-1"] \arrow[d] \arrow[dr, phantom, "\mathrm{Z}"] & & d \arrow[r] \arrow[dr, phantom, "\mathrm{Z}"] & c \arrow[d, dashed, "*"] \\
b \arrow[r, dashed, "*"'] & a^\bullet \arrow[rr, dashed, "*"'] & & d^\bullet
\end{tikzcd}$$

$\leftarrow \cdot \to^* \subseteq \downarrow$ (semi-confluence) $\implies$ ${}^*\!\leftarrow \cdot \to^* \subseteq \downarrow$ (confluence)

Question

how to find suitable bullet function $\bullet$ for CL ?

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Definition

functions $\diamond$ and $\star$ on CL–terms:

$$t^\diamond = \begin{cases} u^\diamond \star v^\diamond & \text{if } t = u\,v \\ t & \text{otherwise} \end{cases} \qquad s \star t = \begin{cases} t & \text{if } s = \mathsf{I} \\ u & \text{if } s = \mathsf{K}\,u \\ u\,t\,(v\,t) & \text{if } s = \mathsf{S}\,u\,v \\ s\,t & \text{otherwise} \end{cases}$$

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Example

$(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond}$

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$$\begin{aligned} (\mathsf{SK(IK)(IIS)})^{\diamond\diamond} &= ((\mathsf{SK(IK)})^\diamond \star (\mathsf{IIS})^\diamond)^\diamond = (((\mathsf{SK})^\diamond \star (\mathsf{IK})^\diamond) \star ((\mathsf{II})^\diamond \star \mathsf{S}^\diamond))^\diamond \\ &= (((\mathsf{S}^\diamond \star \mathsf{K}^\diamond) \star (\mathsf{I}^\diamond \star \mathsf{K}^\diamond)) \star ((\mathsf{I}^\diamond \star \mathsf{I}^\diamond) \star \mathsf{S}))^\diamond = (((\mathsf{S} \star \mathsf{K}) \star (\mathsf{I} \star \mathsf{K})) \star ((\mathsf{I} \star \mathsf{I}) \star \mathsf{S}))^\diamond \\ &= ((\mathsf{SK} \star \mathsf{K}) \star (\mathsf{I} \star \mathsf{S}))^\diamond = (\mathsf{SKK} \star \mathsf{S})^\diamond \end{aligned}$$

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Example (cont'd)

$(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond}$ is common reduct of IIS and $\mathsf{SKK}(\mathsf{IS})$

$$\mathsf{IIS} \leftarrow \mathsf{K}(\mathsf{IIS})(\mathsf{IK}(\mathsf{IIS})) \leftarrow \mathsf{SK}(\mathsf{IK})(\mathsf{IIS}) \rightarrow \mathsf{SKK}(\mathsf{IIS}) \rightarrow \mathsf{SKK}(\mathsf{IS})$$

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CL has Z property for $\diamond$

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Proof (sketch)

for all CL–terms s, t, u, v

① $s\,t \rightarrow^{=} s \star t$

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③ $s \rightarrow^{*} t$ and $u \rightarrow^{*} v \quad \Longrightarrow \quad s \star u \rightarrow^{*} t \star v$

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③ $s \rightarrow^{*} t$ and $u \rightarrow^{*} v \implies s \star u \rightarrow^{*} t \star v$

④ $s \rightarrow^{=} t \implies t \rightarrow^{*} s^{\diamond} \rightarrow^{*} t^{\diamond}$

Outline

Lemma

CL–representable functions are closed under primitive recursion

Proof

$$
\begin{aligned}
f(0, y_1, \dots, y_n) &= g(y_1, \dots, y_n) \\
f(x+1, y_1, \dots, y_n) &= h(f(x, y_1, \dots, y_n), x, y_1, \dots, y_n)
\end{aligned}
$$

with G, H representing g, h

$$
\begin{aligned}
F\,x\,y_1, \dots, y_n &= (\mathsf{zero?}\ x)\ (G\ y_1\ \cdots\ y_n)\ (H\ (F\ (\mathsf{P}\ x)\ y_1\ \cdots\ y_n)\ (\mathsf{P}\ x)\ y_1\ \cdots\ y_n) \\
F &= \mathsf{Y}\ (\langle f\,x\,y_1\ \dots\ y_n\rangle (\mathsf{zero?}\ x)\ (G\ y_1\ \cdots\ y_n)\ (H\ (f\ (\mathsf{P}\ x)\ y_1\ \cdots\ y_n)\ (\mathsf{P}\ x)\ y_1\ \cdots\ y_n))
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\end{aligned}
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Observation

Y has no normal form

Definition

recursion combinator is combinator R such that

$$R\,x\,y\,\underline{0} \leftrightarrow^* x \qquad\qquad R\,x\,y\,\underline{n+1} \leftrightarrow^* y\,\underline{n}\,(R\,x\,y\,\underline{n})$$

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if R is recursion combinator then

$$F = \langle z\,y_1\,\ldots\,y_n\rangle(R\,(G\,y_1\,\cdots\,y_n)\,\langle u\,v\rangle(H\,v\,u\,y_1\,\cdots\,y_n)\,z)$$

represents primitive recursive function f based on g and h

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Proof

$$F\,\underline{0}\,\vec{y} \rightarrow^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{0}$$

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$$F\,\underline{0}\,\vec{y} \to^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{0} \leftrightarrow^* G\,\vec{y}$$

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$$F\,\underline{0}\,\vec{y} \to^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{0} \leftrightarrow^* G\,\vec{y}$$

$$F\,\underline{m+1}\,\vec{y} \to^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m+1}$$

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$$F\,\underline{0}\,\vec{y} \rightarrow^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{0} \leftrightarrow^* G\,\vec{y}$$

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recursion combinator is combinator R such that

$$R\,x\,y\,\underline{0} \leftrightarrow^* x \qquad\qquad R\,x\,y\,\underline{n+1} \leftrightarrow^* y\,\underline{n}\,(R\,x\,y\,\underline{n})$$

Lemma

if R is recursion combinator then

$$F = \langle z\,y_1\,\ldots\,y_n\rangle(R\,(G\,y_1\cdots y_n)\,\langle u\,v\rangle(H\,v\,u\,y_1\cdots y_n)\,z)$$

represents primitive recursive function f based on g and h

Proof

$$\begin{aligned}
F\,\underline{0}\,\vec{y} &\to^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{0} \leftrightarrow^* G\,\vec{y} \\
F\,\underline{m+1}\,\vec{y} &\to^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m+1} \leftrightarrow^* \langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m}\,(R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m}) \\
&\to^* H\,(R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m})\,\underline{m}\,\vec{y}
\end{aligned}$$

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Proof

$$F\,\underline{0}\,\vec{y} \to^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{0} \leftrightarrow^* G\,\vec{y}$$

$$\begin{aligned} F\,\underline{m+1}\,\vec{y} &\to^* R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m+1} \leftrightarrow^* \langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m}\,(R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m}) \\ &\to^* H\,(R\,(G\,\vec{y})\,\langle u\,v\rangle(H\,v\,u\,\vec{y})\,\underline{m})\,\underline{m}\,\vec{y} \leftrightarrow^* H\,(F\,\underline{m}\,\vec{y})\,\underline{m}\,\vec{y} \end{aligned}$$

Definition

$\mathsf{D} = \langle x\, y\, z \rangle (z\, (\mathsf{K}\, y)\, x) = \mathsf{C}(\mathsf{BC}(\mathsf{B}(\mathsf{CI})\mathsf{K}))$

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Lemmata

1. $\mathsf{D}\,x\,y\,\underline{0} \to^{+} x$

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Proof

1. $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0}$

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1. $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z \rangle (z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0} \to^{+} \underline{0}\,(\mathsf{K}\,y)\,x$

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1. $\mathsf{D}\,x\,y\,\underline{0} \to^+ x$

Proof

1. $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0} \to^+ \underline{0}\,(\mathsf{K}\,y)\,x \to^+ x$

Definition

$\mathsf{D} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x) = \mathsf{C}(\mathsf{BC}(\mathsf{B}(\mathsf{CI})\mathsf{K}))$ pairing combinator

Lemmata

1. $\mathsf{D}\,x\,y\,\underline{0} \to^+ x$
2. $\mathsf{D}\,x\,y\,\underline{n} \to^+ y$ for all $n > 0$

Proof

1. $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0} \to^+ \underline{0}\,(\mathsf{K}\,y)\,x \to^+ x$

Definition

$\mathsf{D} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x) = \mathsf{C}(\mathsf{BC}(\mathsf{B}(\mathsf{CI})\mathsf{K}))$ pairing combinator

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Proof

1. $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0} \to^+ \underline{0}\,(\mathsf{K}\,y)\,x \to^+ x$
2. $\mathsf{D}\,x\,y\,\underline{n} \to^* \underline{n}\,(\mathsf{K}\,y)\,x$

Definition

$\mathsf{D} = \langle x\,y\,z\rangle (z\,(\mathsf{K}\,y)\,x) = \mathsf{C}(\mathsf{BC}(\mathsf{B}(\mathsf{CI})\mathsf{K}))$ pairing combinator

Lemmata

1. $\mathsf{D}\,x\,y\,\underline{0} \rightarrow^+ x$
2. $\mathsf{D}\,x\,y\,\underline{n} \rightarrow^+ y$ for all $n > 0$

Proof

1. $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z\rangle (z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0} \rightarrow^+ \underline{0}\,(\mathsf{K}\,y)\,x \rightarrow^+ x$
2. $\mathsf{D}\,x\,y\,\underline{n} \rightarrow^* \underline{n}\,(\mathsf{K}\,y)\,x = \mathsf{S}\,\mathsf{B}\,\underline{n-1}\,(\mathsf{K}\,y)\,x$

Definition

$\mathsf{D} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x) = \mathsf{C}(\mathsf{BC}(\mathsf{B}(\mathsf{CI})\mathsf{K}))$ pairing combinator

Lemmata

1. $\mathsf{D}\,x\,y\,\underline{0} \to^{+} x$
2. $\mathsf{D}\,x\,y\,\underline{n} \to^{+} y$ for all $n > 0$

Proof

1. $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0} \to^{+} \underline{0}\,(\mathsf{K}\,y)\,x \to^{+} x$
2. $\mathsf{D}\,x\,y\,\underline{n} \to^{*} \underline{n}\,(\mathsf{K}\,y)\,x = \mathsf{S}\,\mathsf{B}\,\underline{n-1}\,(\mathsf{K}\,y)\,x$

   $\to \mathsf{B}\,(\mathsf{K}\,y)\,(\underline{n-1}\,(\mathsf{K}\,y))\,x$

Definition

$\mathsf{D} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x) = \mathsf{C}(\mathsf{BC}(\mathsf{B}(\mathsf{CI})\mathsf{K}))$ pairing combinator

Lemmata

1. $\mathsf{D}\;x\;y\;\underline{0} \to^+ x$
2. $\mathsf{D}\;x\;y\;\underline{n} \to^+ y$ for all $n > 0$

Proof

① $\mathsf{D}\;x\;y\;\underline{0} = \langle x\,y\,z\rangle(z\,(\mathsf{K}\,y)\,x)\;x\;y\;\underline{0} \to^+ \underline{0}\;(\mathsf{K}\,y)\;x \to^+ x$

② $$\begin{aligned} \mathsf{D}\;x\;y\;\underline{n} &\to^* \underline{n}\;(\mathsf{K}\,y)\;x = \mathsf{S}\,\mathsf{B}\;\underline{n-1}\;(\mathsf{K}\,y)\;x \\ &\to \mathsf{B}\;(\mathsf{K}\,y)\;(\underline{n-1}\;(\mathsf{K}\,y))\;x \\ &\to^* \mathsf{K}\;y\;(\underline{n-1}\;(\mathsf{K}\,y)\;x) \end{aligned}$$

Definition

$D = \langle x\,y\,z\rangle(z\,(K\,y)\,x) = C(BC(B(CI)K))$ pairing combinator

Lemmata

1. $D\,x\,y\,\underline{0} \to^+ x$
2. $D\,x\,y\,\underline{n} \to^+ y$ for all $n > 0$

Proof

① $D\,x\,y\,\underline{0} = \langle x\,y\,z\rangle(z\,(K\,y)\,x)\,x\,y\,\underline{0} \to^+ \underline{0}\,(K\,y)\,x \to^+ x$

②
$$\begin{aligned} D\,x\,y\,\underline{n} &\to^* \underline{n}\,(K\,y)\,x = S\,B\,\underline{n-1}\,(K\,y)\,x \\ &\to B\,(K\,y)\,(\underline{n-1}\,(K\,y))\,x \\ &\to^* K\,y\,(\underline{n-1}\,(K\,y)\,x) \\ &\to y \end{aligned}$$

Definition

$$\mathsf{Q} = \langle x\, y \rangle (\mathsf{D}\ (\mathsf{succ}\ (y\ \underline{0}))\ (x\ (y\ \underline{0})\ (y\ \underline{1})))$$

Definition

$$\mathsf{Q} = \langle x\, y \rangle (\mathsf{D}\ (\mathsf{succ}\ (y\ \underline{0}))\ (x\ (y\ \underline{0})\ (y\ \underline{1})))$$

Lemmata

1. $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \to^{+} \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$

Definition

$\mathsf{Q} = \langle x\, y \rangle (\mathsf{D}\ (\mathsf{succ}\ (y\ \underline{0}))\ (x\ (y\ \underline{0})\ (y\ \underline{1})))$

Lemmata

1. $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \rightarrow^{+} \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$

Proof

1. $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \rightarrow^{+} \mathsf{D}\ (\mathsf{succ}\ (\mathsf{D}\ \underline{n}\ y\ \underline{0}))\ (x\ (\mathsf{D}\ \underline{n}\ y\ \underline{0})\ (\mathsf{D}\ \underline{n}\ y\ \underline{1}))$

Definition

$$\mathsf{Q} = \langle x\,y \rangle (\mathsf{D}\ (\mathsf{succ}\ (y\ \underline{0}))\ (x\ (y\ \underline{0})\ (y\ \underline{1})))$$

Lemmata

1. $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \to^{+} \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$

Proof

1. $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \to^{+} \mathsf{D}\ (\mathsf{succ}\ (\mathsf{D}\ \underline{n}\ y\ \underline{0}))\ (x\ (\mathsf{D}\ \underline{n}\ y\ \underline{0})\ (\mathsf{D}\ \underline{n}\ y\ \underline{1})) \to^{+} \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$

Definition

$\mathsf{Q} = \langle x\, y \rangle (\mathsf{D}\ (\mathsf{succ}\ (y\ \underline{0}))\ (x\ (y\ \underline{0})\ (y\ \underline{1})))$

Lemmata

1. $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \to^{+} \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$
2. $(\mathsf{Q}\ x)^n\ (\mathsf{D}\ \underline{0}\ y) \to^{+} \mathsf{D}\ \underline{n}\ x_n$ for some term x_n

Proof

① $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \to^{+} \mathsf{D}\ (\mathsf{succ}\ (\mathsf{D}\ \underline{n}\ y\ \underline{0}))\ (x\ (\mathsf{D}\ \underline{n}\ y\ \underline{0})\ (\mathsf{D}\ \underline{n}\ y\ \underline{1})) \to^{+} \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \rightarrow^{+} \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \rightarrow^{*} x^{n}\ y$$

$$(\mathsf{Q}\ y)^{n}\ (\mathsf{D}\ \underline{0}\ x) \rightarrow^{+} \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z \rangle (z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

$$\mathsf{Q}\; y\; (\mathsf{D}\; \underline{n}\; x) \to^{+} \mathsf{D}\; \underline{n+1}\; (y\; \underline{n}\; x) \qquad \underline{n}\; x\; y \to^{*} x^{n}\; y$$

$$(\mathsf{Q}\; y)^{n}\; (\mathsf{D}\; \underline{0}\; x) \to^{+} \mathsf{D}\; \underline{n}\; x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\, y\, z\rangle(z\; (\mathsf{Q}\; y)\; (\mathsf{D}\; \underline{0}\; x)\; \underline{1})$

Lemma

R is recursion combinator

$$\mathsf{Q}\; y\; (\mathsf{D}\; \underline{n}\; x) \to^{+} \mathsf{D}\; \underline{n+1}\; (y\; \underline{n}\; x) \qquad \underline{n}\; x\; y \to^{*} x^{n}\; y$$

$$(\mathsf{Q}\; y)^{n}\; (\mathsf{D}\; \underline{0}\; x) \to^{+} \mathsf{D}\; \underline{n}\; x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\, y\, z\rangle(z\; (\mathsf{Q}\; y)\; (\mathsf{D}\; \underline{0}\; x)\; \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\; x\; y\; \underline{0} \to^{*} \underline{0}\; (\mathsf{Q}\; y)\; (\mathsf{D}\; \underline{0}\; x)\; \underline{1}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad\qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\mathsf{R}\ x\ y\ \underline{n+1} \to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1}$$

$$\mathsf{Q}\; y\; (\mathsf{D}\; \underline{n}\; x) \to^+ \mathsf{D}\; \underline{n+1}\; (y\; \underline{n}\; x) \qquad\qquad \underline{n}\; x\; y \to^* x^n\; y$$

$$(\mathsf{Q}\; y)^n\; (\mathsf{D}\; \underline{0}\; x) \to^+ \mathsf{D}\; \underline{n}\; x_n \quad \text{for some term } x_n$$

Definition

$$\mathsf{R} = \langle x\, y\, z \rangle (z\; (\mathsf{Q}\; y)\; (\mathsf{D}\; \underline{0}\; x)\; \underline{1})$$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\; x\; y\; \underline{0} \to^* \underline{0}\; (\mathsf{Q}\; y)\; (\mathsf{D}\; \underline{0}\; x)\; \underline{1} \to^* \mathsf{D}\; \underline{0}\; x\; \underline{1} \to^* x$$

$$\mathsf{R}\; x\; y\; \underline{n+1} \to^* \underline{n+1}\; (\mathsf{Q}\; y)\; (\mathsf{D}\; \underline{0}\; x)\; \underline{1} \to^* (\mathsf{Q}\; y)^{n+1}\; (\mathsf{D}\; \underline{0}\; x)\; \underline{1}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad\qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\mathsf{R}\ x\ y\ \underline{n+1} \to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\begin{aligned} \mathsf{R}\ x\ y\ \underline{n+1} &\to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1} \\ &\to^* \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \end{aligned}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^{+} \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^{*} x^{n}\ y$$

$$(\mathsf{Q}\ y)^{n}\ (\mathsf{D}\ \underline{0}\ x) \to^{+} \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^{*} \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^{*} \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^{*} x$$

$$\mathsf{R}\ x\ y\ \underline{n+1} \to^{*} \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^{*} (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^{n}\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1}$$

$$\to^{*} \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \to^{*} \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x_n)\ \underline{1}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z \rangle (z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\begin{aligned} \mathsf{R}\ x\ y\ \underline{n+1} &\to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1} \\ &\to^* \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \to^* \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x_n)\ \underline{1} \to^* y\ \underline{n}\ x_n \end{aligned}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\mathsf{R}\ x\ y\ \underline{n+1} \to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1}$$

$$\to^* \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \to^* \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x_n)\ \underline{1} \to^* y\ \underline{n}\ x_n \ {}^*\!\!\leftarrow y\ \underline{n}\ (\mathsf{D}\ \underline{n}\ x_n\ \underline{1})$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\begin{aligned}\mathsf{R}\ x\ y\ \underline{n+1} &\to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1}\\ &\to^* \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \to^* \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x_n)\ \underline{1} \to^* y\ \underline{n}\ x_n\ {}^*\!\!\leftarrow y\ \underline{n}\ (\mathsf{D}\ \underline{n}\ x_n\ \underline{1})\\ &{}^*\!\!\leftarrow y\ \underline{n}\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})\end{aligned}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad\qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\begin{aligned}\mathsf{R}\ x\ y\ \underline{n+1} &\to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1}\\ &\to^* \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \to^* \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x_n)\ \underline{1} \to^* y\ \underline{n}\ x_n\ {}^*\!\!\leftarrow y\ \underline{n}\ (\mathsf{D}\ \underline{n}\ x_n\ \underline{1})\\ &{}^*\!\!\leftarrow y\ \underline{n}\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})\ {}^*\!\!\leftarrow y\ \underline{n}\ (\underline{n}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})\end{aligned}$$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\begin{aligned}
\mathsf{R}\ x\ y\ \underline{n+1} &\to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1} \\
&\to^* \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \to^* \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x_n)\ \underline{1} \to^* y\ \underline{n}\ x_n\ {}^*\!\leftarrow y\ \underline{n}\ (\mathsf{D}\ \underline{n}\ x_n\ \underline{1}) \\
&{}^*\!\leftarrow y\ \underline{n}\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})\ {}^*\!\leftarrow y\ \underline{n}\ (\underline{n}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})\ {}^*\!\leftarrow y\ \underline{n}\ (\mathsf{R}\ x\ y\ \underline{n})
\end{aligned}$$

Remark

$R\ \underline{0}\ K$ represents predecessor function

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CL–representable functions are closed under minimization

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with G representing g

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Outline

Important Concepts

- $\twoheadrightarrow\!\!\!\!\!\!\!\!\shortmid\!\shortmid$
- $\multimap\!\!\!\rightarrow$
- bullet function
- Church–Rosser property
- Church–Rosser theorem
- conversion
- D
- diamond property
- parallel reduction
- pairing combinator
- R
- recursion combinator
- $s \star t$
- $t^{\diamond}$
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homework for November 27