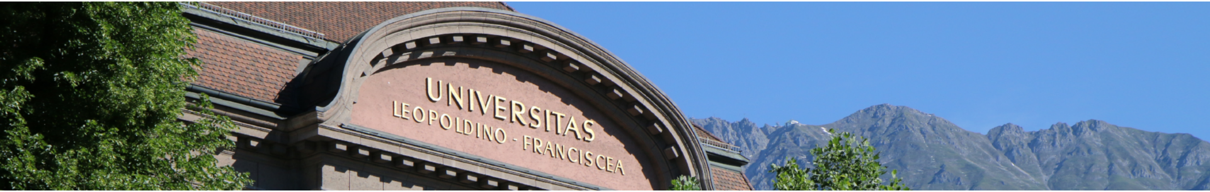
WS 2023 lecture 8



Computability Theory

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Outline

Definitions (Combinatory Logic)

- CL–terms are built from
  - infinite set of variables $\mathcal{V} = \{x, y, z, \ldots\}$
  - constants $\mathsf{I} \quad \mathsf{K} \quad \mathsf{S}$
  - application $s\,t$ for CL–terms s and t
- combinator is CL–term t without variables
- (weak) reduction is smallest relation $\to$ on CL–terms such that

$$\frac{}{\mathsf{I}\,t \to t} \qquad \frac{}{\mathsf{K}\,t\,u \to t} \qquad \frac{}{\mathsf{S}\,t\,u\,v \to t\,v\,(u\,v)} \qquad \frac{t \to u}{t\,v \to u\,v} \qquad \frac{t \to u}{v\,t \to v\,u}$$

  for all CL–terms t, u, v
- normal form is CL–term t such that $t \to u$ for no CL–term u
- $\to^+$ is transitive closure of $\to$
- $\to^*$ is transitive and reflexive closure of $\to$

Definitions

- $t \rightarrow^{!} u$ if $t \rightarrow^{*} u$ for normal form u
- CL–term t is normalizing if $t \rightarrow^{!} u$ for some CL–term u
- infinite reduction is sequence $(t_i)_{i \geqslant 0}$ such that $t_i \rightarrow t_{i+1}$ for all $i \geqslant 0$
- CL–term t is terminating if there are no infinite reductions starting at t
- $\mathsf{B} = \mathsf{S}(\mathsf{KS})\mathsf{K}$ $\qquad \mathsf{C} = \mathsf{S}(\mathsf{BBS})(\mathsf{KK})$ $\qquad \mathsf{Y} = \mathsf{B}(\mathsf{SI})(\mathsf{SII})(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))$
- for all $n \geqslant 0$ Church numeral $\underline{n}$ is combinator $(\mathsf{SB})^n(\mathsf{KI})$
- function $f\colon \mathbb{N}^n \rightarrow \mathbb{N}$ is CL–representable if there exists combinator F such that

$$
\begin{array}{lcl}
f(x_1,\dots,x_n) = y & \Longrightarrow & F\,\underline{x_1}\cdots\underline{x_n} \rightarrow^{*} \underline{y} \\
f(x_1,\dots,x_n) \text{ is undefined} & \Longrightarrow & F\,\underline{x_1}\cdots\underline{x_n} \text{ is not normalizing}
\end{array}
$$

for all $x_1,\dots,x_n,y \in \mathbb{N}$

Lemma

$$\mathsf{B}xyz \rightarrow^{+} x(yz) \qquad \mathsf{C}xyz \rightarrow^{+} xzy \qquad \mathsf{Y}x \rightarrow^{+} x(\mathsf{Y}x)$$

Definition (Bracket Abstraction)

CL–term $[x]t$ is defined for all CL–terms t and variables x:

$$[x]t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}t & \text{if } x \notin \mathcal{V}\mathrm{ar}(t) \\ \mathsf{S}([x]t_1)([x]t_2) & \text{if } t = t_1 t_2 \text{ and } x \in \mathcal{V}\mathrm{ar}(t) \end{cases}$$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$ and $([x]t)\,x \to^* t$ for all CL–terms t and variables x

Corollary (Combinatorial Completeness)

for every CL–term t with $\mathcal{V}\mathrm{ar}(t) = \{x_1, \dots, x_n\}$

1. $\exists$ combinator C such that $C\,x_1 \cdots x_n \to^* t$
2. $\exists$ combinator D such that $D\,x_2 \cdots x_n \to^* t[D/x_1]$

Definition (Bracket Abstraction, Optimized)

- CL–term $\langle x\rangle t$ is defined for all CL–terms t and variables x:

$$\langle x\rangle t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}t & \text{if } x \notin \mathcal{V}\mathrm{ar}(t) \\ u & \text{if } t = u\,x \text{ and } x \notin \mathcal{V}\mathrm{ar}(u) \\ \mathsf{B}u(\langle x\rangle v) & \text{if } t = u\,v \text{ and } x \notin \mathcal{V}\mathrm{ar}(u) \\ \mathsf{C}(\langle x\rangle u)\,v & \text{if } t = u\,v \text{ and } x \notin \mathcal{V}\mathrm{ar}(v) \\ \mathsf{S}(\langle x\rangle u)(\langle x\rangle v) & \text{if } t = u\,v \text{ and } x \in \mathcal{V}\mathrm{ar}(u) \cap \mathcal{V}\mathrm{ar}(v) \end{cases}$$

- $\langle x_1 \ldots x_n\rangle t$ abbreviates $\langle x_1\rangle(\ldots \langle x_n\rangle t \ldots)$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$ and $(\langle x\rangle t)\,x \to^* t$ for all CL–terms t and variables x

Definition

$$\mathsf{T} = \mathsf{K} \qquad \mathsf{F} = \mathsf{KI} \qquad \mathsf{zero?} = \mathsf{C}(\mathsf{B}(\mathsf{CIK}))(\mathsf{K}(\mathsf{KI}))$$

Lemmata

1. initial functions are CL–representable
2. CL–representable functions are closed under composition and primitive recursion

Theorem

CL is confluent: $\forall s \, \forall t \, \forall u \, \left[s \to^* t \wedge s \to^* u \implies \exists v \, (t \to^* v \wedge u \to^* v) \right]$

Corollary

CL has unique normal forms

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability,
while programs, ...

Part II: Combinatory Logic and Lambda Calculus

α–equivalence, abstraction, arithmetization, β–reduction, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem,
Curry–Howard isomorphism, de Bruijn notation, η–reduction, fixed point theorem, intuitionistic propositional logic, λ–definability, normalization theorem, termination, typing, undecidability, Z property, ...

Outline

Definition (Parallel Reduction)

$$\frac{}{t \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; t} \qquad \langle 1 \rangle$$

for all $t \in \{\mathsf{S}, \mathsf{K}, \mathsf{I}\} \cup \mathcal{V}$

$$\frac{}{\mathsf{I}\, t \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; t} \qquad \frac{}{\mathsf{K}\, t\, u \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; t} \qquad \frac{}{\mathsf{S}\, t\, u\, v \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; t\, v\, (u\, v)} \qquad \langle 2 \rangle$$

for all CL–terms t, u, v

$$\frac{t_1 \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; u_1 \qquad t_2 \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; u_2}{t_1\, t_2 \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid\; u_1\, u_2} \qquad \langle 3 \rangle$$

for all CL–terms t_1, t_2, u_1, u_2

Lemma

$\to \;\subseteq\; \twoheadrightarrow\!\!\!\!\!\!\shortmid\shortmid \;\subseteq\; \to^*$

Example

$\mathsf{K(IK)(IK(KSI))} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{KK(KS)}$:

$$\dfrac{\dfrac{\dfrac{}{\mathsf{K} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{K}} \qquad \dfrac{}{\mathsf{IK} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{K}}}{\mathsf{K(IK)} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{KK}} \qquad \dfrac{\dfrac{}{\mathsf{IK} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{K}} \qquad \dfrac{}{\mathsf{KSI} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{S}}}{\mathsf{IK(KSI)} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{KS}}}{\mathsf{K(IK)(IK(KSI))} \mathrel{\rightarrow\!\!\!\!\!\!\Vert}\; \mathsf{KK(KS)}}$$

Lemma

parallel reduction has diamond property: $\forall$ terms s, t, u $\exists$ term v

$$\begin{tikzcd} & s \arrow[dl, "\Vert"'] \arrow[dr, "\Vert"] & \\ t \arrow[dr, dotted, "\Vert"'] & & u \arrow[dl, dotted, "\Vert"] \\ & v & \end{tikzcd}$$

Lemma

$$\forall s\, \forall t\, \forall u \left[\, s \twoheadrightarrow\!\!\!\!+ \; t \wedge s \twoheadrightarrow\!\!\!\!+ \; u \quad \Longrightarrow \quad \exists v\, (t \twoheadrightarrow\!\!\!\!+ \; v \wedge u \twoheadrightarrow\!\!\!\!+ \; v)\,\right]$$

Proof

- induction on derivation of $s \twoheadrightarrow\!\!\!\!+ \; t$ and $s \twoheadrightarrow\!\!\!\!+ \; u$
- easy cases: $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle} t$ or $s \twoheadrightarrow\!\!\!\!+^{\langle 1\rangle} u$ or both $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} u$
  or both $s \twoheadrightarrow\!\!\!\!+^{\langle 3\rangle} t$ and $s \twoheadrightarrow\!\!\!\!+^{\langle 3\rangle} u$
- interesting case (modulo symmetry): $s \twoheadrightarrow\!\!\!\!+^{\langle 2\rangle} t$ and $s \twoheadrightarrow\!\!\!\!+^{\langle 3\rangle} u$

  $s = s_1 s_2$ and $u = u_1 u_2$ with $s_1 \twoheadrightarrow\!\!\!\!+ \; u_1$ and $s_2 \twoheadrightarrow\!\!\!\!+ \; u_2$

  ① $s_1 = \mathsf{I}$ $\quad t = s_2$ $\quad u_1 = \mathsf{I}$ $\quad u \twoheadrightarrow\!\!\!\!+ \; u_2$ $\quad t \twoheadrightarrow\!\!\!\!+ \; u_2$

  ② $s_1 = \mathsf{K}\, s'$ $\quad t = s'$ $\quad u_1 = \mathsf{K}\, u'$ with $s' \twoheadrightarrow\!\!\!\!+ \; u'$ $\quad u \twoheadrightarrow\!\!\!\!+ \; u'$ $\quad t \twoheadrightarrow\!\!\!\!+ \; u'$

  ③ $s_1 = \mathsf{S}\, s'\, s''$ $\quad t = s' s_2\, (s'' s_2)$ $\quad u_1 = \mathsf{S}\, u'\, u''$ with $s' \twoheadrightarrow\!\!\!\!+ \; u'$ and $s'' \twoheadrightarrow\!\!\!\!+ \; u''$
  $u \twoheadrightarrow\!\!\!\!+ \; u' u_2\, (u'' u_2)$ $\quad t \twoheadrightarrow\!\!\!\!+ \; u' u_2\, (u'' u_2)$

Corollary

CL is confluent

Proof

$$
\begin{tikzcd}
\cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[ddd, "*"] \\
\cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \\
\cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\|" marking] \arrow[d, "=" marking] & \cdot \\
\cdot \arrow[rrrr, "*"'] & \cdot & \cdot & \cdot & \cdot
\end{tikzcd}
$$

Definition (Conversion)

$\leftrightarrow^*$ is transitive, reflexive and symmetric closure of $\rightarrow$

Church–Rosser Theorem

CL has Church–Rosser property: $\forall\, t\, \forall\, u\, \big[\, t \leftrightarrow^* u \implies \exists\, v\, (t \rightarrow^* v \,\wedge\, u \rightarrow^* v)\, \big]$

Proof

easy consequence of confluence

Outline

Definition

ARS $\mathcal{A} = \langle A, \to \rangle$ has **Z property** if

$$a \to b \quad \Longrightarrow \quad b \to^* \bullet(a) \to^* \bullet(b)$$

for some function $\bullet$ on A

$$\begin{tikzcd} a \arrow[r] & b \arrow[dl, dotted, "*"'] \\ a^\bullet \arrow[r, dotted, "*"] & b^\bullet \end{tikzcd}$$

Notation

$a^\bullet$ for $\bullet(a)$

Example

ARS

$$\begin{tikzcd} \mathsf{d} \arrow[rrrr, bend left] & \mathsf{a} \arrow[l] \arrow[r] & \mathsf{b} \arrow[r] & \mathsf{c} \arrow[r] & \mathsf{e} \arrow[loop right] \end{tikzcd}$$

- define $\mathsf{a}^\bullet = \mathsf{b}^\bullet = \mathsf{c}^\bullet = \mathsf{d}^\bullet = \mathsf{e}^\bullet = \mathsf{e}$
- every element rewrites to e $\quad \Longrightarrow \quad$ Z property is trivially satisfied

Lemma (Monotonicity)

$a \to^* b \quad \Longrightarrow \quad a^\bullet \to^* b^\bullet$ for every ARS $\langle A, \to \rangle$ with Z property for $\bullet$

Proof

induction on number of steps in $a \to^* b$

Lemma

every ARS with Z property is confluent

Proof

$b \leftarrow a \rightarrow^{n} c \quad \Longrightarrow \quad b \downarrow c$ by induction on n:

- $n = 0 \quad \Longrightarrow \quad c = a \rightarrow b$
- $n > 0 \quad \Longrightarrow \quad a \rightarrow^{n-1} d \rightarrow c$ for some element d

$$\begin{tikzcd}
a \arrow[rr, "n-1"] \arrow[d] \arrow[dr, phantom, "\mathrm{Z}"] & & d \arrow[r] \arrow[dr, phantom, "\mathrm{Z}"] & c \arrow[d, dashed, "*"] \\
b \arrow[r, dashed, "*"'] & a^{\bullet} \arrow[rr, dashed, "*"'] & & d^{\bullet}
\end{tikzcd}$$

$\leftarrow \cdot \rightarrow^{*} \subseteq \downarrow$ (semi-confluence) $\quad \Longrightarrow \quad {}^{*}\!\leftarrow \cdot \rightarrow^{*} \subseteq \downarrow$ (confluence)

Question

how to find suitable bullet function $\bullet$ for CL ?

Definition

functions $\diamond$ and $\star$ on CL–terms:

$$t^\diamond = \begin{cases} u^\diamond \star v^\diamond & \text{if } t = u\,v \\ t & \text{otherwise} \end{cases} \qquad s \star t = \begin{cases} t & \text{if } s = \mathsf{I} \\ u & \text{if } s = \mathsf{K}\,u \\ u\,t\,(v\,t) & \text{if } s = \mathsf{S}\,u\,v \\ st & \text{otherwise} \end{cases}$$

Example

$$\begin{aligned}(\mathsf{SK(IK)(IIS)})^{\diamond\diamond} &= ((\mathsf{SK(IK)})^\diamond \star (\mathsf{IIS})^\diamond)^\diamond = (((\mathsf{SK})^\diamond \star (\mathsf{IK})^\diamond) \star ((\mathsf{II})^\diamond \star \mathsf{S}^\diamond))^\diamond \\ &= (((\mathsf{S}^\diamond \star \mathsf{K}^\diamond) \star (\mathsf{I}^\diamond \star \mathsf{K}^\diamond)) \star ((\mathsf{I}^\diamond \star \mathsf{I}^\diamond) \star \mathsf{S}))^\diamond = (((\mathsf{S} \star \mathsf{K}) \star (\mathsf{I} \star \mathsf{K})) \star ((\mathsf{I} \star \mathsf{I}) \star \mathsf{S}))^\diamond \\ &= ((\mathsf{SK} \star \mathsf{K}) \star (\mathsf{I} \star \mathsf{S}))^\diamond = (\mathsf{SKK} \star \mathsf{S})^\diamond = (\mathsf{KS(KS)})^\diamond = \mathsf{KS} \star \mathsf{KS} = \mathsf{S}\end{aligned}$$

Example (cont'd)

$(\mathsf{SK}(\mathsf{IK})(\mathsf{IIS}))^{\diamond\diamond}$ is common reduct of IIS and $\mathsf{SKK}(\mathsf{IS})$

$$\mathsf{IIS} \leftarrow \mathsf{K}(\mathsf{IIS})(\mathsf{IK}(\mathsf{IIS})) \leftarrow \mathsf{SK}(\mathsf{IK})(\mathsf{IIS}) \rightarrow \mathsf{SKK}(\mathsf{IIS}) \rightarrow \mathsf{SKK}(\mathsf{IS})$$

Theorem

CL has Z property for $\diamond$

Proof (sketch)

for all CL–terms s, t, u, v

① $s\,t \rightarrow^{=} s \star t$

② $t \rightarrow^{*} t^{\diamond}$

③ $s \rightarrow^{*} t$ and $u \rightarrow^{*} v \implies s \star u \rightarrow^{*} t \star v$

④ $s \rightarrow^{=} t \implies t \rightarrow^{*} s^{\diamond} \rightarrow^{*} t^{\diamond}$

Outline

Lemma

CL–representable functions are closed under primitive recursion

Proof

$$\begin{aligned} f(0, y_1, \ldots, y_n) &= g(y_1, \ldots, y_n) \\ f(x+1, y_1, \ldots, y_n) &= h(f(x, y_1, \ldots, y_n), x, y_1, \ldots, y_n) \end{aligned}$$

with G, H representing g, h

$$\begin{aligned} F\, x\, y_1, \ldots, y_n &= (\mathsf{zero?}\ x)\ (G\, y_1 \cdots y_n)\ (H\ (F\ (\mathsf{P}\ x)\ y_1 \cdots y_n)\ (\mathsf{P}\ x)\ y_1 \cdots y_n) \\ F &= \mathsf{Y}\ (\langle f\, x\, y_1\ \ldots\ y_n \rangle (\mathsf{zero?}\ x)\ (G\, y_1 \cdots y_n)\ (H\ (f\ (\mathsf{P}\ x)\ y_1 \cdots y_n)\ (\mathsf{P}\ x)\ y_1 \cdots y_n)) \end{aligned}$$

Observation

Y has no normal form

Definition

recursion combinator is combinator R such that

$$R\ x\ y\ \underline{0} \leftrightarrow^* x \qquad\qquad R\ x\ y\ \underline{n+1} \leftrightarrow^* y\ \underline{n}\ (R\ x\ y\ \underline{n})$$

Lemma

if R is recursion combinator then

$$F = \langle z\, y_1\, \dots\, y_n \rangle (R\ (G\ y_1 \cdots y_n)\ \langle u\, v \rangle (H\ v\ u\ y_1 \cdots y_n)\ z)$$

represents primitive recursive function f based on g and h

Proof

$$F\ \underline{0}\ \vec{y} \to^* R\ (G\ \vec{y})\ \langle u\, v \rangle (H\ v\ u\ \vec{y})\ \underline{0} \leftrightarrow^* G\ \vec{y}$$

$$\begin{aligned} F\ \underline{m+1}\ \vec{y} &\to^* R\ (G\ \vec{y})\ \langle u\, v \rangle (H\ v\ u\ \vec{y})\ \underline{m+1} \leftrightarrow^* \langle u\, v \rangle (H\ v\ u\ \vec{y})\ \underline{m}\ (R\ (G\ \vec{y})\ \langle u\, v \rangle (H\ v\ u\ \vec{y})\ \underline{m}) \\ &\to^* H\ (R\ (G\ \vec{y})\ \langle u\, v \rangle (H\ v\ u\ \vec{y})\ \underline{m})\ \underline{m}\ \vec{y} \leftrightarrow^* H\,(F\ \underline{m}\ \vec{y})\ \underline{m}\ \vec{y} \end{aligned}$$

Definition

$\mathsf{D} = \langle x\,y\,z \rangle (z\,(\mathsf{K}\,y)\,x) = \mathsf{C}(\mathsf{BC}(\mathsf{B}(\mathsf{CI})\mathsf{K}))$ pairing combinator

Lemmata

1. $\mathsf{D}\,x\,y\,\underline{0} \to^{+} x$
2. $\mathsf{D}\,x\,y\,\underline{n} \to^{+} y$ for all $n > 0$

Proof

① $\mathsf{D}\,x\,y\,\underline{0} = \langle x\,y\,z \rangle (z\,(\mathsf{K}\,y)\,x)\,x\,y\,\underline{0} \to^{+} \underline{0}\,(\mathsf{K}\,y)\,x \to^{+} x$

②
$$\begin{aligned}
\mathsf{D}\,x\,y\,\underline{n} &\to^{*} \underline{n}\,(\mathsf{K}\,y)\,x = \mathsf{S}\,\mathsf{B}\,\underline{n-1}\,(\mathsf{K}\,y)\,x \\
&\to \mathsf{B}\,(\mathsf{K}\,y)\,(\underline{n-1}\,(\mathsf{K}\,y))\,x \\
&\to^{*} \mathsf{K}\,y\,(\underline{n-1}\,(\mathsf{K}\,y)\,x) \\
&\to y
\end{aligned}$$

Definition

$\mathsf{Q} = \langle x\,y \rangle (\mathsf{D}\ (\mathsf{succ}\ (y\ \underline{0}))\ (x\ (y\ \underline{0})\ (y\ \underline{1})))$

Lemmata

1. $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \to^+ \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$
2. $(\mathsf{Q}\ x)^n\ (\mathsf{D}\ \underline{0}\ y) \to^+ \mathsf{D}\ \underline{n}\ x_n$ for some term x_n

Proof

① $\mathsf{Q}\ x\ (\mathsf{D}\ \underline{n}\ y) \to^+ \mathsf{D}\ (\mathsf{succ}\ (\mathsf{D}\ \underline{n}\ y\ \underline{0}))\ (x\ (\mathsf{D}\ \underline{n}\ y\ \underline{0})\ (\mathsf{D}\ \underline{n}\ y\ \underline{1})) \to^+ \mathsf{D}\ \underline{n+1}\ (x\ \underline{n}\ y)$

$$\mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x) \to^+ \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x) \qquad \underline{n}\ x\ y \to^* x^n\ y$$

$$(\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x) \to^+ \mathsf{D}\ \underline{n}\ x_n \quad \text{for some term } x_n$$

Definition

$$\mathsf{R} = \langle x\,y\,z\rangle(z\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1})$$

Lemma

R is recursion combinator

Proof

$$\mathsf{R}\ x\ y\ \underline{0} \to^* \underline{0}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* \mathsf{D}\ \underline{0}\ x\ \underline{1} \to^* x$$

$$\begin{aligned}
\mathsf{R}\ x\ y\ \underline{n+1} &\to^* \underline{n+1}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} \to^* (\mathsf{Q}\ y)^{n+1}\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1} = \mathsf{Q}\ y\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x))\ \underline{1} \\
&\to^* \mathsf{Q}\ y\ (\mathsf{D}\ \underline{n}\ x_n)\ \underline{1} \to^* \mathsf{D}\ \underline{n+1}\ (y\ \underline{n}\ x_n)\ \underline{1} \to^* y\ \underline{n}\ x_n \ {}^*\!\!\leftarrow y\ \underline{n}\ (\mathsf{D}\ \underline{n}\ x_n\ \underline{1}) \\
&{}^*\!\!\leftarrow y\ \underline{n}\ ((\mathsf{Q}\ y)^n\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1}) \ {}^*\!\!\leftarrow y\ \underline{n}\ (\underline{n}\ (\mathsf{Q}\ y)\ (\mathsf{D}\ \underline{0}\ x)\ \underline{1}) \ {}^*\!\!\leftarrow y\ \underline{n}\ (\mathsf{R}\ x\ y\ \underline{n})
\end{aligned}$$

Remark

$\mathsf{R}\ \underline{0}\ \mathsf{K}$ represents predecessor function

Lemma

CL–representable functions are closed under minimization

Proof

$$f(x_1,\ldots,x_n) = (\mu\, i)\,(g(i,x_1,\ldots,x_n) = 0)$$

with G representing g

$$F = H\,\underline{0}$$

with

$$\begin{aligned} H &= \langle i\,x_1\,\cdots\,x_n\rangle(\mathsf{zero?}\,(G\,i\,x_1\,\cdots\,x_n)\,i\,(H\,(\mathsf{succ}\,i)\,x_1\,\cdots\,x_n)) \\ &= \mathsf{Y}\,(\langle h\,i\,x_1\,\cdots\,x_n\rangle(\mathsf{zero?}\,(G\,i\,x_1\,\cdots\,x_n)\,i\,(h\,(\mathsf{succ}\,i)\,x_1\,\cdots\,x_n))) \end{aligned}$$

Theorem

partial recursive functions are CL–representable ?

Problem

- partial recursive function

$$f(x) = z((\mu\, i)\ (x + i = 0))$$

  is undefined for $x > 0$

- combinator (produced by construction in previous proof)

$$F = \langle x\rangle(\mathsf{zero}\ (M\ x)) \quad \text{with} \quad M = \mathsf{Y}\ (\langle h\, i\, x\rangle(\mathsf{zero?}\ (\mathsf{add}\ x\ i)\ i\ (h\ (\mathsf{succ}\ i)\ x)))\ \underline{0}$$

  satisfies

$$F\ \underline{x} \to^{+} \mathsf{zero}\ (M\ \underline{x}) = \mathsf{K}\ (\mathsf{KI})\ (M\ \underline{x}) \to \mathsf{KI} = \underline{0}$$

  for all $x \geqslant 0$

Outline

Important Concepts

- $-\!\!\Vert\!\!\rightarrow$
- $-\!\!\bullet\!\!\rightarrow$
- bullet function
- Church–Rosser property
- Church–Rosser theorem
- conversion
- D
- diamond property
- parallel reduction
- pairing combinator
- R
- recursion combinator
- $s \star t$
- $t^{\diamond}$
- Z property

homework for November 27