universität
innsbruck

WS 2023 lecture 10

Computability Theory

Aart Middeldorp

Outline

Definitions

- (many-step) strategy $\mathcal{S}$ for ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is relation $\rightarrow_{\mathcal{S}}$ on A such that $\rightarrow_{\mathcal{S}} \subseteq \rightarrow^{+}$ and $\mathsf{NF}(\rightarrow_{\mathcal{S}}) = \mathsf{NF}(\mathcal{A})$
- one-step strategy satisfies $\rightarrow_{\mathcal{S}} \subseteq \rightarrow$
- strategy $\mathcal{S}$ is deterministic if $a = b$ whenever $a \; {}_{\mathcal{S}}{\leftarrow} \cdot \rightarrow_{\mathcal{S}} b$
- strategy $\mathcal{S}$ for ARS $\mathcal{A}$ is normalizing if every normalizing element is $\mathcal{S}$–terminating
- strategy $\mathcal{S}$ for ARS $\mathcal{A}$ is hyper–normalizing if every normalizing element is terminating with respect to $\rightarrow^{*} \cdot \rightarrow_{\mathcal{S}} \cdot \rightarrow^{*}$
- strategy $\mathcal{S}_{\bullet}$ for ARS $\mathcal{A}$ with Z property for $\bullet$: $a \mathrel{\bullet\!\!\rightarrow} b$ if $a \notin \mathsf{NF}(\mathcal{A})$ and $b = a^{\bullet}$
- root reduction $\xrightarrow{\epsilon}$: $\quad \mathsf{I}\,t \xrightarrow{\epsilon} t \qquad \mathsf{K}\,t\,u \xrightarrow{\epsilon} t \qquad \mathsf{S}\,t\,u\,v \xrightarrow{\epsilon} t\,v\,(u\,v)$
- leftmost outermost reduction $\xrightarrow{\mathsf{lo}}$:

$$\frac{t \xrightarrow{\epsilon} u}{t \xrightarrow{\mathsf{lo}} u} \qquad \frac{t \xrightarrow{\mathsf{lo}} u \qquad t\,v \in \mathsf{NF}(\xrightarrow{\epsilon})}{t\,v \xrightarrow{\mathsf{lo}} u\,v} \qquad \frac{t \xrightarrow{\mathsf{lo}} u \qquad v\,t \in \mathsf{NF}(\xrightarrow{\epsilon}) \qquad v \in \mathsf{NF}(\rightarrow)}{v\,t \xrightarrow{\mathsf{lo}} v\,u}$$

- $t \xrightarrow{\neg\mathsf{lo}} u$ if $t \rightarrow u$ but not $t \xrightarrow{\mathsf{lo}} u$

Theorem

$\mathcal{S}_\bullet$ is hyper−normalizing for every ARS with Z property for $\bullet$

Theorem (Factorization)

$\rightarrow^* \subseteq \xrightarrow{\text{lo}}{}^* \cdot \xrightarrow{\neg\text{lo}}{}^*$

Normalization Theorem

leftmost outermost reduction is hyper−normalizing

Theorem

partial recursive functions are CL−representable by combinators in normal form

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course−of−values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s−m−n theorem, total recursive functions, undecidability, while programs, ...

Part II: Combinatory Logic and Lambda Calculus

α−equivalence, abstraction, arithmetization, β−reduction, CL−representability, combinators, combinatorial completeness, Church numerals, Church−Rosser theorem, Curry−Howard isomorphism, de Bruijn notation, η−reduction, fixed point theorem, intuitionistic propositional logic, λ−definability, normalization theorem, termination, typing, undecidability, Z property, ...

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function φ is partial recursive $\iff$ φ is CL–representable

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The main theorem of this chapter will be that every partial recursive function can be represented in both λ and CL.

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The main theorem of this chapter will be that every partial recursive function can be represented in both λ and CL.

The converse is also true, that every function representable in λ or CL is partial recursive. But its proof is too boring to include in this book.

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Gödel number of CL–term is defined inductively:

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predicates $\mathsf{I}(x)$, $\mathsf{K}(x)$, $\mathsf{S}(x)$, $\mathsf{A}(x)$, $\mathsf{V}(x)$, $\mathsf{term}(x)$ are defined inductively:

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$$\mathsf{S}(x) \iff \mathsf{seq}(x) \wedge \mathsf{len}(x) = 1 \wedge (x)_1 = 2$$

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\mathsf{S}(x) &\iff \mathsf{seq}(x) \wedge \mathsf{len}(x) = 1 \wedge (x)_1 = 2 \\
\mathsf{A}(x) &\iff \mathsf{triple}(x) \wedge (x)_1 = 3 \wedge \mathsf{term}((x)_2) \wedge \mathsf{term}((x)_3)
\end{aligned}$$

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\mathsf{A}(x) &\iff \mathsf{triple}(x) \wedge (x)_1 = 3 \wedge \mathsf{term}((x)_2) \wedge \mathsf{term}((x)_3) \\
\mathsf{V}(x) &\iff \mathsf{seq}(x) \wedge \mathsf{len}(x) = 2 \wedge (x)_1 = 4
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\mathsf{S}(x) &\iff \mathsf{seq}(x) \wedge \mathsf{len}(x) = 1 \wedge (x)_1 = 2 \\
\mathsf{A}(x) &\iff \mathsf{triple}(x) \wedge (x)_1 = 3 \wedge \mathsf{term}((x)_2) \wedge \mathsf{term}((x)_3) \\
\mathsf{V}(x) &\iff \mathsf{seq}(x) \wedge \mathsf{len}(x) = 2 \wedge (x)_1 = 4 \\
\mathsf{term}(x) &\iff \mathsf{I}(x) \vee \mathsf{K}(x) \vee \mathsf{S}(x) \vee \mathsf{A}(x) \vee \mathsf{V}(x)
\end{aligned}$$

$$\frac{}{\mathsf{I}\,t \to t} \qquad \frac{}{\mathsf{K}\,t\,u \to t} \qquad \frac{}{\mathsf{S}\,t\,u\,v \to t\,v\,(u\,v)} \qquad \frac{t \to u}{t\,v \to u\,v} \qquad \frac{t \to u}{v\,t \to v\,u}$$

Definition

predicate $\mathrm{step}(x, y)$ is inductively defined:

$$\mathrm{step}(x, y) \iff \mathrm{term}(x) \land \mathrm{term}(y) \land \mathrm{A}(x) \land$$

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$$\mathsf{step}(x, y) \iff \mathsf{term}(x) \wedge \mathsf{term}(y) \wedge \mathsf{A}(x) \wedge$$
$$\Big[\; [\mathsf{I}((x)_2) \wedge (x)_3 = y] \; \Big.$$

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$$\begin{aligned}\mathsf{step}(x,y) \quad \Longleftrightarrow \quad & \mathsf{term}(x) \wedge \mathsf{term}(y) \wedge \mathsf{A}(x) \wedge \\ & \Big[\ \big[\mathsf{I}((x)_2) \wedge (x)_3 = y\big] \\ & \vee \big[\mathsf{A}((x)_2) \wedge \mathsf{K}((x)_{2,2}) \wedge (x)_{2,3} = y\big]\end{aligned}$$

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& \vee \big[\mathrm{A}((x)_2) \wedge \mathrm{K}((x)_{2,2}) \wedge (x)_{2,3} = y \big] \\
& \vee \big[\mathrm{A}((x)_2) \wedge \mathrm{A}((x)_{2,2}) \wedge \mathrm{S}((x)_{2,2,2}) \wedge \mathrm{A}(y) \wedge \mathrm{A}((y)_2) \wedge \mathrm{A}((y)_3) \wedge \\
& \quad (x)_{2,2,3} = (y)_{2,2} \wedge (x)_{2,3} = (y)_{3,2} \wedge (x)_3 = (y)_{2,3} \wedge (x)_3 = (y)_{3,3} \big]
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& \vee \big[\mathrm{A}(y) \wedge \mathrm{step}((x)_2,(y)_2) \wedge (x)_3 = (y)_3\big]
\end{aligned}$$

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& \vee \big[\mathrm{A}(y) \wedge \mathrm{step}((x)_2, (y)_2) \wedge (x)_3 = (y)_3\big] \\
& \vee \big[\mathrm{A}(y) \wedge (x)_2 = (y)_2 \wedge \mathrm{step}((x)_3, (y)_3)\big]\ \Big]
\end{aligned}$$

Definitions

- predicates reduction(x) and conversion(x)

$$\mathrm{reduction}(x) \iff \mathrm{seq}(x) \wedge (\forall\, i < \mathrm{len}(x) \dot{-} 1)\left[\,\mathrm{step}((x)_i, (x)_{i+1})\,\right]$$

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- $\mathrm{enc}(n) = \mathfrak{g}(\underline{n})$

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Example

$\text{enc}(0) = \mathfrak{g}(\text{KI})$

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$$\mathsf{enc}(0) = \mathfrak{g}(\mathsf{KI}) = \langle 3, \langle 1 \rangle, \langle 0 \rangle \rangle$$

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$\mathsf{enc}(0) = \mathfrak{g}(\mathsf{KI}) = \langle 3, \langle 1 \rangle, \langle 0 \rangle \rangle = 18375000$

Definition

- function dec : $\mathbb{N} \to \mathbb{N}$

$$\mathrm{dec}(x) = \begin{cases} 0 & \text{if } \mathrm{zero}(x) \\ \mathrm{dec}((x)_3) + 1 & \text{if } \mathrm{numeral}(x) \wedge \neg\mathrm{zero}(x) \\ 0 & \text{otherwise} \end{cases}$$

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CL–representable functions are partial recursive

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Theorem

CL–representable functions are partial recursive

Proof

$$\mathsf{first}(x) = (x)_1 \quad \mathsf{last}(x) = (x)_{\mathsf{len}(x)}$$

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CL–representable functions are partial recursive

Proof

$$\text{first}(x) = (x)_1 \quad \text{last}(x) = (x)_{\text{len}(x)}$$

$$\text{reduction}(i) \wedge \text{first}(i) = \mathfrak{g}(F\,\underline{x_1} \cdots \underline{x_n}) \wedge \text{numeral}(\text{last}(i))$$

Definition

- function $\text{dec} \colon \mathbb{N} \to \mathbb{N}$

$$\text{dec}(x) = \begin{cases} 0 & \text{if } \text{zero}(x) \\ \text{dec}((x)_3) + 1 & \text{if } \text{numeral}(x) \wedge \neg\text{zero}(x) \\ 0 & \text{otherwise} \end{cases}$$

Theorem

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$$\mathrm{first}(x) = (x)_1 \quad \mathrm{last}(x) = (x)_{\mathrm{len}(x)} \quad \mathfrak{g}(F\,\underline{x_1}\,\cdots\,\underline{x_n}) = \langle 3, \ldots \langle 3, \mathfrak{g}(F), \mathrm{enc}(x_1)\rangle, \ldots \mathrm{enc}(x_n)\rangle$$

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$$f(x_1, \dots, x_n) \simeq \mathsf{dec}(\mathsf{last}((\mu\, i)\,[\mathsf{reduction}(i) \wedge \mathsf{first}(i) = \mathfrak{g}(F\,\underline{x_1}\,\cdots\,\underline{x_n}) \wedge \mathsf{numeral}(\mathsf{last}(i))]))$$

Outline

Notation

$\ulcorner t \urcorner = \underline{\mathfrak{g}(t)}$ is Church numeral of Gödel number of CL−term t

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$\forall$ CL–term F $\exists$ CL–term X such that $F \ulcorner X \urcorner \leftrightarrow^* X$

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Outline

Definitions

- sets T and U of CL–terms are **recursively separable** if $\{\mathfrak{g}(t) \mid t \in T\}$ and $\{\mathfrak{g}(u) \mid u \in U\}$ are recursively separable

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Theorem

non-empty conversion–closed sets of CL–terms are recursively inseparable

Proof (by contradiction)

- non-empty conversion–closed sets T and U of CL–terms
- $\exists$ recursive function $f \colon \mathbb{N} \to \{0,1\}$ such that

$$t \in T \implies f(\mathfrak{g}(t)) = 0 \qquad\qquad t \in U \implies f(\mathfrak{g}(t)) = 1$$

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- $V = \{t \mid f(\mathfrak{g}(t)) = 0\}$

Proof (cont'd)

- $T \subseteq V$ and $U \cap V = \varnothing$

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- $A \in T$ and $B \in U$

Proof (cont'd)

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$$t \in V \quad \Longrightarrow \quad G\,\ulcorner t \urcorner \leftrightarrow^* B \qquad\qquad t \notin V \quad \Longrightarrow \quad G\,\ulcorner t \urcorner \leftrightarrow^* A$$

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- $\exists\, X$ such that $G\,\ulcorner X \urcorner \leftrightarrow^* X$ by fixed point theorem

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$$X \in V$$
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$$X \notin V \implies X \leftrightarrow^* G\,\ulcorner X \urcorner \leftrightarrow^* A \implies X \in T \implies X \in V$$

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$\lightning$

Theorem

non-trivial conversion−closed sets of CL−terms are not recursive

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Proof

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- $\sim T = \{t \mid t \notin T\}$

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Proof

- non-trivial conversion−closed set T of CL−terms
- $\sim T = \{t \mid t \notin T\}$ is non-empty conversion−closed set of CL−terms

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Proof

- non-trivial conversion–closed set T of CL–terms
- $\sim T = \{t \mid t \notin T\}$ is non-empty conversion–closed set of CL–terms
- T and $\sim T$ are recursively inseparable

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Corollary

set of normalizing CL−terms is not recursive

Theorem

non-trivial conversion–closed sets of CL–terms are not recursive

Proof

- non-trivial conversion–closed set T of CL–terms
- $\sim T = \{ t \mid t \notin T \}$ is non-empty conversion–closed set of CL–terms
- T and $\sim T$ are recursively inseparable $\implies$ T is not recursive

Corollary

set of normalizing CL–terms is not recursive: decision problem

instance: CL–term t

question: is t normalizing ?

is undecidable

Outline

Definition (Types)

- infinite set $\mathbb{V}$ of type variables

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  - if $\sigma, \tau \in \mathbb{T}$ then $(\sigma \to \tau) \in \mathbb{T}$

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Notation

- outermost parentheses are omitted
- $\to$ is right-associative: $\rho \to \sigma \to \tau$ stands for $\rho \to (\sigma \to \tau)$

Definition (Type Assignment, Curry-style)

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for all types σ

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for all types σ, τ, ρ

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$$\frac{t : \sigma \to \tau \qquad u : \sigma}{t\,u : \tau}$$

for all types σ, τ, ρ and CL–terms t and u

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for all types σ, τ, ρ and CL–terms t and u

Notation

$\Gamma \vdash t : \tau$ if $t : \tau$ can be derived in TA from assumptions in Γ

Example ❶

$\vdash$ SKK $: \sigma \to \sigma$ for all types σ

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$\vdash$ SKK : $\sigma \to \sigma$ for all types σ

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Example ❶

$\vdash$ $\mathsf{SKK} : \sigma \to \sigma$ for all types σ

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Example ❷

$x : \sigma \to \tau,\ y : \sigma \vdash \mathsf{K}x\mathsf{I}y : \tau$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{K} : (\sigma \to \tau) \to (\rho \to \rho) \to \sigma \to \tau} \quad x : \sigma \to \tau}{\mathsf{K}x : (\rho \to \rho) \to \sigma \to \tau} \quad \dfrac{}{\mathsf{I} : \rho \to \rho}}{\mathsf{K}x\mathsf{I} : \sigma \to \tau} \quad y : \sigma}{\mathsf{K}x\mathsf{I}y : \tau}$$

Theorem

if $\Gamma, x : \sigma \vdash t : \tau$ and $x \notin \mathcal{V}\mathrm{ar}(\Gamma)$ then $\Gamma \vdash [x]t : \sigma \to \tau$

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Proof

induction on definition of $[x]t$

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① $t = x$

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① $t = x \quad \Longrightarrow \quad [x]t = \mathsf{I}$

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② $x \notin \mathcal{V}\mathrm{ar}(t) \quad \Longrightarrow \quad [x]t = \mathsf{K}\,t$

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③ $t = t_1 t_2 \implies [x]t = \mathsf{S}\,([x]t_1)\,([x]t_2)$

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if $\Gamma, x : \sigma \vdash t : \tau$ and $x \notin \mathcal{V}\mathrm{ar}(\Gamma)$ then $\Gamma \vdash [x]t : \sigma \to \tau$

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if $\Gamma, x : \sigma \vdash t : \tau$ and $x \notin \mathcal{V}\mathrm{ar}(\Gamma)$ then $\Gamma \vdash [x]t : \sigma \to \tau$

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$\Gamma \vdash [x]t_1 : \sigma \to \tau_1 \to \tau$ and $\Gamma \vdash [x]t_2 : \sigma \to \tau_1$ by induction hypothesis

Theorem

if $\Gamma, x : \sigma \vdash t : \tau$ and $x \notin \mathcal{V}\mathrm{ar}(\Gamma)$ then $\Gamma \vdash [x]t : \sigma \to \tau$

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$\vdash \mathsf{S} : (\sigma \to \tau_1 \to \tau) \to (\sigma \to \tau_1) \to \sigma \to \tau$

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if $\Gamma, x : \sigma \vdash t : \tau$ and $x \notin \mathcal{V}\mathrm{ar}(\Gamma)$ then $\Gamma \vdash [x]t : \sigma \to \tau$

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$\Gamma \vdash [x]t_1 : \sigma \to \tau_1 \to \tau$ and $\Gamma \vdash [x]t_2 : \sigma \to \tau_1$ by induction hypothesis

$\vdash \mathsf{S} : (\sigma \to \tau_1 \to \tau) \to (\sigma \to \tau_1) \to \sigma \to \tau \implies \Gamma \vdash \mathsf{S}([x]t_1)([x]t_2) : \sigma \to \tau$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

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① $t = \mathsf{I}\, t_1 \to t_1 = u \quad \Longrightarrow \quad \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma$

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② $t = \mathsf{K}\, t_1 t_2 \to t_1 = u$

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② $t = \mathsf{K}\, t_1 t_2 \to t_1 = u \implies \Gamma \vdash \mathsf{K} t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

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$\implies \vdash \mathsf{K} : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \rho$

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Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

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③ $t = \mathsf{S}\, t_1 t_2 t_3 \to t_1 t_3 (t_2 t_3) = u$

Theorem (Subject Reduction)

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Proof

① $t = \mathsf{I}\, t_1 \to t_1 = u \implies \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \implies \sigma = \tau \implies \Gamma \vdash u : \tau$

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③ $t = \mathsf{S}\, t_1 t_2 t_3 \to t_1 t_3 (t_2 t_3) = u \implies \Gamma \vdash \mathsf{S}\, t_1 t_2 : \sigma \to \tau$ and $\Gamma \vdash t_3 : \sigma$

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③ $t = \mathsf{S}\, t_1 t_2 t_3 \to t_1 t_3 (t_2 t_3) = u \implies \Gamma \vdash \mathsf{S}\, t_1 t_2 : \sigma \to \tau$ and $\Gamma \vdash t_3 : \sigma$

$\implies \Gamma \vdash \mathsf{S}\, t_1 : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_2 : \rho \implies \vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \mu$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof

① $t = \mathsf{I}\, t_1 \to t_1 = u \implies \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \implies \sigma = \tau \implies \Gamma \vdash u : \tau$

② $t = \mathsf{K}\, t_1 t_2 \to t_1 = u \implies \Gamma \vdash \mathsf{K} t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

$\implies \vdash \mathsf{K} : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \rho \implies \rho = \tau \implies \Gamma \vdash u : \tau$

③ $t = \mathsf{S}\, t_1 t_2 t_3 \to t_1 t_3 (t_2 t_3) = u \implies \Gamma \vdash \mathsf{S}\, t_1 t_2 : \sigma \to \tau$ and $\Gamma \vdash t_3 : \sigma$

$\implies \Gamma \vdash \mathsf{S}\, t_1 : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_2 : \rho \implies \vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \mu$

$\implies \rho = \sigma \to \rho_1$ and $\mu = \sigma \to \rho_1 \to \tau$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof

① $t = \mathsf{I}\,t_1 \to t_1 = u \implies \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \implies \sigma = \tau \implies \Gamma \vdash u : \tau$

② $t = \mathsf{K}\,t_1 t_2 \to t_1 = u \implies \Gamma \vdash \mathsf{K} t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

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$\implies \Gamma \vdash \mathsf{S}\,t_1 : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_2 : \rho \implies \vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \mu$

$\implies \rho = \sigma \to \rho_1$ and $\mu = \sigma \to \rho_1 \to \tau \implies \Gamma \vdash t_1 t_3 : \rho_1 \to \tau$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof

① $t = \mathsf{I}\, t_1 \to t_1 = u \quad\Longrightarrow\quad \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \quad\Longrightarrow\quad \sigma = \tau \quad\Longrightarrow\quad \Gamma \vdash u : \tau$

② $t = \mathsf{K}\, t_1 t_2 \to t_1 = u \quad\Longrightarrow\quad \Gamma \vdash \mathsf{K} t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

$\Longrightarrow\quad \vdash \mathsf{K} : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \rho \quad\Longrightarrow\quad \rho = \tau \quad\Longrightarrow\quad \Gamma \vdash u : \tau$

③ $t = \mathsf{S}\, t_1 t_2 t_3 \to t_1 t_3 (t_2 t_3) = u \quad\Longrightarrow\quad \Gamma \vdash \mathsf{S}\, t_1 t_2 : \sigma \to \tau$ and $\Gamma \vdash t_3 : \sigma$

$\Longrightarrow\quad \Gamma \vdash \mathsf{S}\, t_1 : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_2 : \rho \quad\Longrightarrow\quad \vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \mu$

$\Longrightarrow\quad \rho = \sigma \to \rho_1$ and $\mu = \sigma \to \rho_1 \to \tau \quad\Longrightarrow\quad \Gamma \vdash t_1 t_3 : \rho_1 \to \tau$ and $\Gamma \vdash t_2 t_3 : \rho_1$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof

① $t = \mathsf{I}\, t_1 \to t_1 = u \implies \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \implies \sigma = \tau \implies \Gamma \vdash u : \tau$

② $t = \mathsf{K}\, t_1 t_2 \to t_1 = u \implies \Gamma \vdash \mathsf{K} t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

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$\implies \Gamma \vdash \mathsf{S}\, t_1 : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_2 : \rho \implies \vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \mu$

$\implies \rho = \sigma \to \rho_1$ and $\mu = \sigma \to \rho_1 \to \tau \implies \Gamma \vdash t_1 t_3 : \rho_1 \to \tau$ and $\Gamma \vdash t_2 t_3 : \rho_1$

$\implies \Gamma \vdash u : \tau$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof

① $t = \mathsf{I}\, t_1 \to t_1 = u \implies \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \implies \sigma = \tau \implies \Gamma \vdash u : \tau$

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$\implies \Gamma \vdash \mathsf{S}\, t_1 : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_2 : \rho \implies \vdash \mathsf{S} : \mu \to \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \mu$

$\implies \rho = \sigma \to \rho_1$ and $\mu = \sigma \to \rho_1 \to \tau \implies \Gamma \vdash t_1 t_3 : \rho_1 \to \tau$ and $\Gamma \vdash t_2 t_3 : \rho_1$

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④ $t = t_1 t_2 \to u_1 t_2 = u$ with $t_1 \to u_1$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof

① $t = \mathsf{I}\, t_1 \to t_1 = u \implies \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \implies \sigma = \tau \implies \Gamma \vdash u : \tau$

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④ $t = t_1 t_2 \to u_1 t_2 = u$ with $t_1 \to u_1 \implies \Gamma \vdash t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof

① $t = \mathsf{I}\, t_1 \to t_1 = u \implies \vdash \mathsf{I} : \sigma \to \tau$ and $\Gamma \vdash t_1 : \sigma \implies \sigma = \tau \implies \Gamma \vdash u : \tau$

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$\implies \Gamma \vdash u_1 : \sigma \to \tau$ by induction hypothesis

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

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$\implies \vdash \mathsf{K} : \rho \to \sigma \to \tau$ and $\Gamma \vdash t_1 : \rho \implies \rho = \tau \implies \Gamma \vdash u : \tau$

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$\implies \Gamma \vdash u_1 : \sigma \to \tau$ by induction hypothesis $\implies \Gamma \vdash u : \tau$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof (cont'd)

⑤ $t = t_1 t_2 \to t_1 u_2 = u$ with $t_2 \to u_2$

Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof (cont'd)

⑤ $t = t_1 t_2 \to t_1 u_2 = u$ with $t_2 \to u_2 \quad \Longrightarrow \quad \Gamma \vdash t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

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$\implies \Gamma \vdash u_2 : \sigma$ by induction hypothesis

Theorem (Subject Reduction)

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Proof (cont'd)

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Theorem (Subject Reduction)

if $\Gamma \vdash t : \tau$ and $t \to u$ then $\Gamma \vdash u : \tau$

Proof (cont'd)

⑤ $t = t_1 t_2 \to t_1 u_2 = u$ with $t_2 \to u_2 \quad \Longrightarrow \quad \Gamma \vdash t_1 : \sigma \to \tau$ and $\Gamma \vdash t_2 : \sigma$

$\Longrightarrow \quad \Gamma \vdash u_2 : \sigma$ by induction hypothesis $\quad \Longrightarrow \quad \Gamma \vdash u : \tau$

Definition

CL–term t with $\mathcal{V}\mathrm{ar}(t) = \{x_1, \ldots, x_n\}$ is typable if

$$x_1 : \rho_1, \ldots, x_n : \rho_n \vdash t : \tau$$

for some types $\rho_1, \ldots, \rho_n, \tau$

Outline

Important Concepts

- $\ulcorner t \urcorner$
- arithmetization
- $\mathbb{C}$
- conversion−closed
- $\mathrm{dec}(n)$
- $\mathrm{enc}(n)$
- $\Gamma \vdash t : \tau$
- $\mathfrak{g}(t)$
- Gödel number
- recursive separability
- SN
- subject reduction
- $\mathbb{T}$
- TA
- type
- type assignment
- type constant
- $\mathbb{V}$

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homework for December 11