universität innsbruck

WS 2023 lecture 12

Computability Theory

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Outline

Theorem (Strong Normalization)

typable CL–terms are terminating (SN)

Definition

typable CL–term t is strongly computable (SC) if

- t has atomic type $\tau \in \mathbb{V} \cup \mathbb{C}$ and is SN
- t has type $\sigma \to \tau$ and $t\,u$ is SC whenever $u : \sigma$ is SC

Lemma

every typable term is SC and every SC term is terminating

Theorem

problem

instance: CL–term t

question: is t typable ?

is decidable

Definition

principal type of combinator t is any type σ such that

① $\vdash t : \sigma$

② if $\vdash t : \tau$ then τ is substitution instance of σ

Theorem

every typable combinator has principal type

Type Inference

principle types can be computed by typing rules of TA (with type variables σ, τ, ρ)

$$\frac{}{\mathsf{I} : \sigma \to \sigma} \qquad \frac{}{\mathsf{K} : \sigma \to \tau \to \sigma} \qquad \frac{}{\mathsf{S} : (\rho \to \sigma \to \tau) \to (\rho \to \sigma) \to \rho \to \tau} \qquad \frac{t : \sigma \to \tau \qquad u : \sigma}{t\,u : \tau}$$

and unification algorithm

Definition

type τ is inhabited if $\vdash t : \tau$ for some combinator t

Theorem

problem

instance: type τ

question: is τ inhabited ?

is decidable

Intuitionistic Propositional Logic

- basic connectives $\rightarrow \quad \wedge \quad \vee \quad \bot$
- derived connectives
  - $\neg\varphi$ abbreviates $\varphi \rightarrow \bot$
  - $\top$ abbreviates $\bot \rightarrow \bot$
  - $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- implication fragment contains only $\rightarrow$

Definition

Kripke model is triple $\mathcal{C} = \langle C, \leqslant, \Vdash \rangle$ with

- non-empty set C of states
- partial order $\leqslant$ on C
- binary relation $\Vdash$ between elements of C and propositional atoms

such that $d \Vdash p$ whenever $c \Vdash p$ and $c \leqslant d$

Definition

Kripke model $\mathcal{C} = \langle C, \leqslant, \Vdash \rangle$, $c \in C$

- $c \Vdash \varphi \wedge \psi$ if and only if $c \Vdash \varphi$ and $c \Vdash \psi$
- $c \Vdash \varphi \vee \psi$ if and only if $c \Vdash \varphi$ or $c \Vdash \psi$
- $c \Vdash \varphi \rightarrow \psi$ if and only if $d \Vdash \psi$ for all $d \geqslant c$ with $d \Vdash \varphi$
- $c \nVdash \bot$

Terminology

c forces p if $c \Vdash p$

Definition

Kripke model $\mathcal{C} = \langle C, \leqslant, \Vdash \rangle$, $c \in C$

- $c \Vdash \Gamma$ if $c \Vdash \varphi$ for all $\varphi \in \Gamma$
- $\mathcal{C} \Vdash \varphi$ if $c \Vdash \varphi$ for all $c \in C$

Definition

$\Gamma \Vdash \varphi$ if $c \Vdash \varphi$ whenever $c \Vdash \Gamma$ for all Kripke models $\mathcal{C} = \langle C, \leqslant, \Vdash \rangle$ and $c \in C$

Lemma (Monotonicity)

if $c \leqslant d$ and $c \Vdash \varphi$ then $d \Vdash \varphi$

Lemma

if $\Vdash \varphi \vee \psi$ then $\Vdash \varphi$ or $\Vdash \psi$

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course−of−values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s−m−n theorem, total recursive functions, undecidability, while programs, ...

Part II: Combinatory Logic and Lambda Calculus

α−equivalence, abstraction, arithmetization, β−reduction, CL−representability, combinators, combinatorial completeness, Church numerals, Church−Rosser theorem, Curry−Howard isomorphism, de Bruijn notation, η−reduction, fixed point theorem, intuitionistic propositional logic, λ−definability, normalization theorem, termination, typing, undecidability, Z property, ...

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Outline

1. Summary of Previous Lecture
2. **Evaluation**
3. Hilbert Systems
4. Curry–Howard Isomorphism
5. Intuitionistic Propositional Logic
6. Summary

Online Evaluation in Presence

https://lv-analyse.uibk.ac.at/evasys/public/online/index

Definition

Hilbert system (for implication fragment) consists of two axioms and modus ponens:

$$\frac{}{\varphi \to \psi \to \varphi} \qquad \frac{}{(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi} \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}$$

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Definitions

- derivation in Hilbert system from set Γ of formulas is finite sequence of formulas such that each formula is
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- proof in Hilbert system is derivation from $\varnothing$

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Definitions

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  - axiom or
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- φ is consequence of set Γ ($\Gamma \vdash_h \varphi$) if φ is last line of derivation from Γ
- proof in Hilbert system is derivation from $\varnothing$
- formula φ is theorem ($\vdash_h \varphi$) if φ is consequence of $\varnothing$

Example

$\varphi \rightarrow \varphi$ is theorem:

1 $(\varphi \rightarrow (\varphi \rightarrow \varphi) \rightarrow \varphi) \rightarrow (\varphi \rightarrow \varphi \rightarrow \varphi) \rightarrow \varphi \rightarrow \varphi$ axiom

Example

$\varphi \to \varphi$ is theorem:

1	$(\varphi \to (\varphi \to \varphi) \to \varphi) \to (\varphi \to \varphi \to \varphi) \to \varphi \to \varphi$	axiom
2	$\varphi \to (\varphi \to \varphi) \to \varphi$	axiom

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3	$(\varphi \to \varphi \to \varphi) \to \varphi \to \varphi$	modus ponens 1, 2

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Deduction Theorem

$$\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \psi \quad \iff \quad \Gamma \vdash_{\mathsf{h}} \varphi \to \psi$$

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Proof ($\Longleftarrow$)

- suppose $\Gamma \vdash_{\mathsf{h}} \varphi \to \psi$

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- $\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \varphi \to \psi$

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Proof ($\Longleftarrow$)

- suppose $\Gamma \vdash_{\mathsf{h}} \varphi \to \psi$
- $\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \varphi \to \psi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \varphi$

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Deduction Theorem

$$\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \psi \iff \Gamma \vdash_{\mathsf{h}} \varphi \to \psi$$

Proof ($\Longleftarrow$)

- suppose $\Gamma \vdash_{\mathsf{h}} \varphi \to \psi$
- $\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \varphi \to \psi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \varphi \implies \Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \psi$ by modus ponens

Example

$(\varphi \to \psi \to \chi) \to \psi \to \varphi \to \chi$ is theorem

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$(\varphi \to \psi \to \chi) \to \psi \to \varphi \to \chi$ is theorem:

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- $\{\varphi \to \psi \to \chi, \psi\} \vdash_{\mathsf{h}} \varphi \to \chi$ by deduction theorem

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- $\{\varphi \to \psi \to \chi, \psi\} \vdash_{\mathsf{h}} \varphi \to \chi$ by deduction theorem
- $\{\varphi \to \psi \to \chi\} \vdash_{\mathsf{h}} \psi \to \varphi \to \chi$ by deduction theorem
- $\vdash_{\mathsf{h}} (\varphi \to \psi \to \chi) \to \psi \to \varphi \to \chi$ by deduction theorem

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- suppose $\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \psi$

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- let $\Pi_1 : \chi_1, \ldots, \chi_n$ be derivation of ψ from $\Gamma \cup \{\varphi\}$, so $\chi_n = \psi$

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- consider new sequence $\Pi_2 : \varphi \to \chi_1, \dots, \varphi \to \chi_n$

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- insert extra lines into Π_2 and use modus ponens, as follows:

① if χ_i is axiom or member of Γ

② if $\chi_i = \varphi$

③ if χ_i is derived with modus ponens from χ_j and χ_k with $j, k < i$

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  insert χ_i and $\chi_i \to \varphi \to \chi_i$ before $\varphi \to \chi_i$

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③ if χ_i is derived with modus ponens from χ_j and χ_k with $j, k < i$ then $\chi_k = (\chi_j \to \chi_i)$

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  ③ if χ_i is derived with modus ponens from χ_j and χ_k with $j, k < i$ then $\chi_k = (\chi_j \to \chi_i)$
  insert $(\varphi \to \chi_j \to \chi_i) \to (\varphi \to \chi_j) \to \varphi \to \chi_i$ and $(\varphi \to \chi_j) \to \varphi \to \chi_i$ before $\varphi \to \chi_i$

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- resulting sequence is derivation of $\varphi \to \psi$ from Γ

Theorem

Hilbert system is sound and complete with respect to Kripke models for implication fragment:

$$\Gamma \vdash_{\mathsf{h}} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash \varphi$$

Proof ($\Longrightarrow$)

suppose $\Gamma \vdash_h \varphi$

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Proof ($\Longrightarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$, we prove $\Gamma \Vdash \varphi$ by induction on length of derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$:

- $\varphi \in \Gamma$
- $\varphi = (\psi_1 \to \psi_2 \to \psi_1)$
- $\varphi = ((\psi_1 \to \psi_2 \to \psi_3) \to (\psi_1 \to \psi_2) \to \psi_1 \to \psi_3)$
- φ is obtained by modus ponens

Proof ($\Longrightarrow$)

suppose $\Gamma \vdash_h \varphi$, we prove $\Gamma \Vdash \varphi$ by induction on length of derivation of $\Gamma \vdash_h \varphi$:

- $\varphi \in \Gamma$

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- $\varphi = (\psi_1 \to \psi_2 \to \psi_1)$

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  $\Vdash \varphi$ by definition of $\Vdash$

- $\varphi = ((\psi_1 \to \psi_2 \to \psi_3) \to (\psi_1 \to \psi_2) \to \psi_1 \to \psi_3)$

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  $\Vdash \varphi$ by definition of $\Vdash$ and thus also $\Gamma \Vdash \varphi$

- $\varphi = ((\psi_1 \to \psi_2 \to \psi_3) \to (\psi_1 \to \psi_2) \to \psi_1 \to \psi_3)$

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- $\varphi = (\psi_1 \to \psi_2 \to \psi_1)$

  $\Vdash \varphi$ by definition of $\Vdash$ and thus also $\Gamma \Vdash \varphi$

- $\varphi = ((\psi_1 \to \psi_2 \to \psi_3) \to (\psi_1 \to \psi_2) \to \psi_1 \to \psi_3)$

  $\Vdash \varphi$ by definition of $\Vdash$ and thus also $\Gamma \Vdash \varphi$

- φ is obtained by modus ponens

Proof ($\Longrightarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$, we prove $\Gamma \Vdash \varphi$ by induction on length of derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$:

- $\varphi \in \Gamma$

  $\Gamma \Vdash \varphi$ holds trivially

- $\varphi = (\psi_1 \to \psi_2 \to \psi_1)$

  $\Vdash \varphi$ by definition of $\Vdash$ and thus also $\Gamma \Vdash \varphi$

- $\varphi = ((\psi_1 \to \psi_2 \to \psi_3) \to (\psi_1 \to \psi_2) \to \psi_1 \to \psi_3)$

  $\Vdash \varphi$ by definition of $\Vdash$ and thus also $\Gamma \Vdash \varphi$

- φ is obtained by modus ponens

  $\Gamma \vdash_{\mathsf{h}} \psi$ and $\Gamma \vdash_{\mathsf{h}} \psi \to \varphi$

Proof ($\Longrightarrow$)

suppose $\Gamma \vdash_h \varphi$, we prove $\Gamma \Vdash \varphi$ by induction on length of derivation of $\Gamma \vdash_h \varphi$:

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- $\varphi = (\psi_1 \to \psi_2 \to \psi_1)$

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- $\varphi = ((\psi_1 \to \psi_2 \to \psi_3) \to (\psi_1 \to \psi_2) \to \psi_1 \to \psi_3)$

  $\Vdash \varphi$ by definition of $\Vdash$ and thus also $\Gamma \Vdash \varphi$

- φ is obtained by modus ponens

  $\Gamma \vdash_h \psi$ and $\Gamma \vdash_h \psi \to \varphi$ are shorter derivations

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- φ is obtained by modus ponens

  $\Gamma \vdash_h \psi$ and $\Gamma \vdash_h \psi \to \varphi$ are shorter derivations

  $\Gamma \Vdash \psi$ and $\Gamma \Vdash \psi \to \varphi$ by induction hypothesis

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  $\Gamma \Vdash \varphi$ by definition of $\Vdash$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

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define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
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proof of claim (induction on ψ): consider $\psi = (\psi_1 \to \psi_2)$

$\Longrightarrow$ let $\Delta \Vdash \psi$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

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claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

proof of claim (induction on ψ): consider $\psi = (\psi_1 \to \psi_2)$

$\Longrightarrow$ let $\Delta \Vdash \psi$ and define $\Delta' = \{\chi \mid \Delta, \psi_1 \vdash_{\mathsf{h}} \chi\}$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

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$\Longrightarrow$ let $\Delta \Vdash \psi$ and define $\Delta' = \{\chi \mid \Delta, \psi_1 \vdash_{\mathsf{h}} \chi\}$

$\psi_1 \in \Delta'$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
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claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

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$\Longrightarrow$ let $\Delta \Vdash \psi$ and define $\Delta' = \{\chi \mid \Delta, \psi_1 \vdash_{\mathsf{h}} \chi\}$

$\psi_1 \in \Delta' \in C$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_h \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_h \psi\}\}$
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$\Longrightarrow$ let $\Delta \Vdash \psi$ and define $\Delta' = \{\chi \mid \Delta, \psi_1 \vdash_h \chi\}$

$\psi_1 \in \Delta' \in C$ and thus $\Delta' \Vdash \psi_1$ by induction hypothesis

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

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$\Longrightarrow$ let $\Delta \Vdash \psi$ and define $\Delta' = \{\chi \mid \Delta, \psi_1 \vdash_{\mathsf{h}} \chi\}$

$\psi_1 \in \Delta' \in C$ and thus $\Delta' \Vdash \psi_1$ by induction hypothesis

$\Delta' \Vdash \psi_2$ because $\Delta \subseteq \Delta'$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_\mathsf{h} \varphi$ does not hold

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- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_\mathsf{h} \psi\}\}$
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$\psi_1 \in \Delta' \in C$ and thus $\Delta' \Vdash \psi_1$ by induction hypothesis

$\Delta' \Vdash \psi_2$ because $\Delta \subseteq \Delta'$ and thus $\psi_2 \in \Delta'$ by induction hypothesis

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta \text{ and } \Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
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claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

proof of claim (induction on ψ): consider $\psi = (\psi_1 \to \psi_2)$

$\Longrightarrow$ let $\Delta \Vdash \psi$ and define $\Delta' = \{\chi \mid \Delta, \psi_1 \vdash_{\mathsf{h}} \chi\}$

$\psi_1 \in \Delta' \in C$ and thus $\Delta' \Vdash \psi_1$ by induction hypothesis

$\Delta' \Vdash \psi_2$ because $\Delta \subseteq \Delta'$ and thus $\psi_2 \in \Delta'$ by induction hypothesis

$\Delta, \psi_1 \vdash_{\mathsf{h}} \psi_2$

Proof ($\Longleftarrow$)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
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proof of claim (induction on ψ): consider $\psi = (\psi_1 \to \psi_2)$

$\Longrightarrow$ let $\Delta \Vdash \psi$ and define $\Delta' = \{\chi \mid \Delta, \psi_1 \vdash_{\mathsf{h}} \chi\}$

$\psi_1 \in \Delta' \in C$ and thus $\Delta' \Vdash \psi_1$ by induction hypothesis

$\Delta' \Vdash \psi_2$ because $\Delta \subseteq \Delta'$ and thus $\psi_2 \in \Delta'$ by induction hypothesis

$\Delta, \psi_1 \vdash_{\mathsf{h}} \psi_2$

$\Delta \vdash_{\mathsf{h}} \psi$ by deduction theorem

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

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claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

proof of claim: consider $\psi = (\psi_1 \to \psi_2)$

$\Longleftarrow$ let $\psi \in \Delta$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
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proof of claim: consider $\psi = (\psi_1 \to \psi_2)$

$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_h \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_h \psi\}\}$
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$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

$\psi_1 \in \Delta'$ by induction hypothesis

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

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$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

$\psi_1 \in \Delta'$ by induction hypothesis and thus $\Delta' \vdash_{\mathsf{h}} \psi_1$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_h \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_h \psi\}\}$
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$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

$\psi_1 \in \Delta'$ by induction hypothesis and thus $\Delta' \vdash_h \psi_1$

$\Delta' \vdash_h \psi$ because $\Delta \vdash_h \psi$ and $\Delta \subseteq \Delta'$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_h \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

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proof of claim: consider $\psi = (\psi_1 \to \psi_2)$

$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

$\psi_1 \in \Delta'$ by induction hypothesis and thus $\Delta' \vdash_h \psi_1$

$\Delta' \vdash_h \psi$ because $\Delta \vdash_h \psi$ and $\Delta \subseteq \Delta'$

$\Delta' \vdash_h \psi_2$ by modus ponens

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

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$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

$\psi_1 \in \Delta'$ by induction hypothesis and thus $\Delta' \vdash_{\mathsf{h}} \psi_1$

$\Delta' \vdash_{\mathsf{h}} \psi$ because $\Delta \vdash_{\mathsf{h}} \psi$ and $\Delta \subseteq \Delta'$

$\Delta' \vdash_{\mathsf{h}} \psi_2$ by modus ponens and thus $\psi_2 \in \Delta'$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

proof of claim: consider $\psi = (\psi_1 \to \psi_2)$

$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

$\psi_1 \in \Delta'$ by induction hypothesis and thus $\Delta' \vdash_{\mathsf{h}} \psi_1$

$\Delta' \vdash_{\mathsf{h}} \psi$ because $\Delta \vdash_{\mathsf{h}} \psi$ and $\Delta \subseteq \Delta'$

$\Delta' \vdash_{\mathsf{h}} \psi_2$ by modus ponens and thus $\psi_2 \in \Delta'$

$\Delta' \Vdash \psi_2$ by induction hypothesis

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

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proof of claim: consider $\psi = (\psi_1 \to \psi_2)$

$\Longleftarrow$ let $\psi \in \Delta$ and consider state $\Delta' \supseteq \Delta$ with $\Delta' \Vdash \psi_1$

$\psi_1 \in \Delta'$ by induction hypothesis and thus $\Delta' \vdash_{\mathsf{h}} \psi_1$

$\Delta' \vdash_{\mathsf{h}} \psi$ because $\Delta \vdash_{\mathsf{h}} \psi$ and $\Delta \subseteq \Delta'$

$\Delta' \vdash_{\mathsf{h}} \psi_2$ by modus ponens and thus $\psi_2 \in \Delta'$

$\Delta' \Vdash \psi_2$ by induction hypothesis and thus $\Delta' \Vdash \psi$ by definition of $\Vdash$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

define $\Delta = \{\psi \mid \Gamma \vdash_{\mathsf{h}} \psi\}$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

define $\Delta = \{\psi \mid \Gamma \vdash_{\mathsf{h}} \psi\}$

$\Delta \in C$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

define $\Delta = \{\psi \mid \Gamma \vdash_{\mathsf{h}} \psi\}$

$\Delta \in C$ and $\varphi \notin \Delta$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

define $\Delta = \{\psi \mid \Gamma \vdash_{\mathsf{h}} \psi\}$

$\Delta \in C$ and $\varphi \notin \Delta$ and thus $\Gamma \subseteq \Delta$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta \text{ and } \Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

define $\Delta = \{\psi \mid \Gamma \vdash_{\mathsf{h}} \psi\}$

$\Delta \in C$ and $\varphi \notin \Delta$ and thus $\Gamma \subseteq \Delta$ and $\Delta \nVdash \varphi$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_{\mathsf{h}} \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_{\mathsf{h}} \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

define $\Delta = \{\psi \mid \Gamma \vdash_{\mathsf{h}} \psi\}$

$\Delta \in C$ and $\varphi \notin \Delta$ and thus $\Gamma \subseteq \Delta$ and $\Delta \nVdash \varphi$

$\Gamma \nVdash \varphi$ by definition of $\Vdash$

Proof ($\Longleftarrow$, cont'd)

suppose $\Gamma \vdash_h \varphi$ does not hold

define Kripke model $\mathcal{C} = \langle C, \subseteq, \Vdash \rangle$ with

- $C = \{\Delta \mid \Gamma \subseteq \Delta$ and $\Delta = \{\psi \mid \Delta \vdash_h \psi\}\}$
- $\Delta \Vdash p$ if $p \in \Delta$ for propositional atoms p

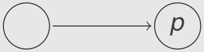
claim: $\Delta \Vdash \psi \iff \psi \in \Delta$ for all $\Delta \in C$ and implicational formulas ψ

define $\Delta = \{\psi \mid \Gamma \vdash_h \psi\}$

$\Delta \in C$ and $\varphi \notin \Delta$ and thus $\Gamma \subseteq \Delta$ and $\Delta \nVdash \varphi$

$\Gamma \nVdash \varphi$ by definition of $\Vdash$

Example (Peirce's Law)

$\nvdash_h ((p \to q) \to p) \to p$ because of Kripke model 

Outline

type assignment

$$\Gamma, x : \tau \vdash x : \tau$$

$$\Gamma \vdash \mathsf{K} : \sigma \to \tau \to \sigma$$

$$\Gamma \vdash \mathsf{S} : (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho$$

$$\frac{\Gamma \vdash t : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash t\,u : \tau}$$

type assignment	Hilbert system
$\Gamma, x : \tau \vdash x : \tau$	$\Gamma, \varphi \vdash \varphi$
$\Gamma \vdash \mathsf{K} : \sigma \to \tau \to \sigma$	$\Gamma \vdash \varphi \to \psi \to \varphi$
$\Gamma \vdash \mathsf{S} : (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho$	$\Gamma \vdash (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
$\dfrac{\Gamma \vdash t : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash t\,u : \tau}$	$\dfrac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$

type assignment	Hilbert system
$\Gamma, \quad \tau \vdash \quad \tau$	$\Gamma, \varphi \vdash \varphi$
$\Gamma \vdash \quad \sigma \to \tau \to \sigma$	$\Gamma \vdash \varphi \to \psi \to \varphi$
$\Gamma \vdash \quad (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho$	$\Gamma \vdash (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
$\dfrac{\Gamma \vdash \quad \sigma \to \tau \qquad \Gamma \vdash \quad \sigma}{\Gamma \vdash \quad \tau}$	$\dfrac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\text{h}} \tau$

type assignment	Hilbert system
$\Gamma, x : \tau \vdash x : \tau$	$\Gamma, \varphi \vdash \varphi$
$\Gamma \vdash \mathsf{K} : \sigma \to \tau \to \sigma$	$\Gamma \vdash \varphi \to \psi \to \varphi$
$\Gamma \vdash \mathsf{S} : (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho$	$\Gamma \vdash (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
$\dfrac{\Gamma \vdash t : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash t\,u : \tau}$	$\dfrac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$
2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\text{types}(\Delta) = \Gamma$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\text{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\text{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \quad \Longrightarrow \quad \text{types}(\Gamma) = \text{types}(\Gamma'), \tau$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\mathrm{types}(\Gamma) \vdash_{\mathrm{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \quad \Longrightarrow \quad \mathrm{types}(\Gamma) = \mathrm{types}(\Gamma'), \tau$ and thus $\mathrm{types}(\Gamma) \vdash_{\mathrm{h}} \tau$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \implies \text{types}(\Gamma) = \text{types}(\Gamma'), \tau$ and thus $\text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$
- $t = \mathsf{K}$ and $\tau = (\sigma \to \rho \to \sigma)$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \quad \Longrightarrow \quad \text{types}(\Gamma) = \text{types}(\Gamma'), \tau$ and thus $\text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$
- $t = \mathsf{K}$ and $\tau = (\sigma \to \rho \to \sigma) \quad \Longrightarrow \quad \text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by axiom K

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_h \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \implies \text{types}(\Gamma) = \text{types}(\Gamma'), \tau$ and thus $\text{types}(\Gamma) \vdash_h \tau$
- $t = \mathsf{K}$ and $\tau = (\sigma \to \rho \to \sigma) \implies \text{types}(\Gamma) \vdash_h \tau$ by axiom K
- $t = \mathsf{S}$ and $\tau = ((\sigma \to \rho \to \chi) \to (\sigma \to \rho) \to \sigma \to \chi$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \implies \mathsf{types}(\Gamma) = \mathsf{types}(\Gamma'), \tau$ and thus $\mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$
- $t = \mathsf{K}$ and $\tau = (\sigma \to \rho \to \sigma) \implies \mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by axiom K
- $t = \mathsf{S}$ and $\tau = ((\sigma \to \rho \to \chi) \to (\sigma \to \rho) \to \sigma \to \chi \implies \mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by axiom S

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \implies \mathsf{types}(\Gamma) = \mathsf{types}(\Gamma'), \tau$ and thus $\mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$
- $t = \mathsf{K}$ and $\tau = (\sigma \to \rho \to \sigma) \implies \mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by axiom K
- $t = \mathsf{S}$ and $\tau = ((\sigma \to \rho \to \chi) \to (\sigma \to \rho) \to \sigma \to \chi \implies \mathsf{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by axiom S
- $t = u\,v$ and $\Gamma \vdash u : \sigma \to \tau$ and $\Gamma \vdash v : \sigma$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\text{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \implies \text{types}(\Gamma) = \text{types}(\Gamma'), \tau$ and thus $\text{types}(\Gamma) \vdash_{\text{h}} \tau$
- $t = \text{K}$ and $\tau = (\sigma \to \rho \to \sigma) \implies \text{types}(\Gamma) \vdash_{\text{h}} \tau$ by axiom K
- $t = \text{S}$ and $\tau = ((\sigma \to \rho \to \chi) \to (\sigma \to \rho) \to \sigma \to \chi \implies \text{types}(\Gamma) \vdash_{\text{h}} \tau$ by axiom S
- $t = u\,v$ and $\Gamma \vdash u : \sigma \to \tau$ and $\Gamma \vdash v : \sigma$

  $\implies \text{types}(\Gamma) \vdash_{\text{h}} \sigma \to \tau$ and $\text{types}(\Gamma) \vdash_{\text{h}} \sigma$ by induction hypothesis

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$

Proof

induction on derivation of judgement $\Gamma \vdash t : \tau$

- $t = x$ and $\Gamma = \Gamma', x : \tau \implies \text{types}(\Gamma) = \text{types}(\Gamma'), \tau$ and thus $\text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$
- $t = \mathsf{K}$ and $\tau = (\sigma \to \rho \to \sigma) \implies \text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by axiom K
- $t = \mathsf{S}$ and $\tau = ((\sigma \to \rho \to \chi) \to (\sigma \to \rho) \to \sigma \to \chi \implies \text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by axiom S
- $t = u\,v$ and $\Gamma \vdash u : \sigma \to \tau$ and $\Gamma \vdash v : \sigma$

  $\implies \text{types}(\Gamma) \vdash_{\mathsf{h}} \sigma \to \tau$ and $\text{types}(\Gamma) \vdash_{\mathsf{h}} \sigma$ by induction hypothesis

  $\implies \text{types}(\Gamma) \vdash_{\mathsf{h}} \tau$ by modus ponens

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathsf{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathsf{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

interesting case: φ is obtained by modus ponens

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathsf{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

interesting case: φ is obtained by modus ponens

$\Gamma \vdash_{\mathsf{h}} \psi \to \varphi$ and $\Gamma \vdash_{\mathsf{h}} \psi$

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\text{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

interesting case: φ is obtained by modus ponens

$\Gamma \vdash_{\mathsf{h}} \psi \to \varphi$ and $\Gamma \vdash_{\mathsf{h}} \psi$

induction hypothesis: $\Delta_1 \vdash t_1 : \psi \to \varphi$ and $\Delta_2 \vdash t_2 : \psi$

for some t_1, Δ_1, t_2, Δ_2 with $\text{types}(\Delta_1) = \text{types}(\Delta_2) = \Gamma$

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathrm{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathrm{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathrm{h}} \varphi$

interesting case: φ is obtained by modus ponens

$\Gamma \vdash_{\mathrm{h}} \psi \to \varphi$ and $\Gamma \vdash_{\mathrm{h}} \psi$

induction hypothesis: $\Delta_1 \vdash t_1 : \psi \to \varphi$ and $\Delta_2 \vdash t_2 : \psi$
for some t_1, Δ_1, t_2, Δ_2 with $\mathrm{types}(\Delta_1) = \mathrm{types}(\Delta_2) = \Gamma$

suppose $\Gamma = \{\chi_1, \ldots, \chi_n\}$

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathsf{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

interesting case: φ is obtained by modus ponens

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suppose $\Gamma = \{\chi_1, \dots, \chi_n\}$

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Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathsf{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

interesting case: φ is obtained by modus ponens

$\Gamma \vdash_{\mathsf{h}} \psi \to \varphi$ and $\Gamma \vdash_{\mathsf{h}} \psi$

induction hypothesis: $\Delta_1 \vdash t_1 : \psi \to \varphi$ and $\Delta_2 \vdash t_2 : \psi$
for some t_1, Δ_1, t_2, Δ_2 with $\mathsf{types}(\Delta_1) = \mathsf{types}(\Delta_2) = \Gamma$

suppose $\Gamma = \{\chi_1, \dots, \chi_n\}$

$\Delta_1 = \{x_1 : \chi_1, \dots, x_n : \chi_n\}$ and $\Delta_2 = \{y_1 : \chi_1, \dots, y_n : \chi_n\}$

let t_2' be obtained from t_2 by replacing every y_i with x_i

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathsf{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

interesting case: φ is obtained by modus ponens

$\Gamma \vdash_{\mathsf{h}} \psi \to \varphi$ and $\Gamma \vdash_{\mathsf{h}} \psi$

induction hypothesis: $\Delta_1 \vdash t_1 : \psi \to \varphi$ and $\Delta_2 \vdash t_2 : \psi$
for some t_1, Δ_1, t_2, Δ_2 with $\mathsf{types}(\Delta_1) = \mathsf{types}(\Delta_2) = \Gamma$

suppose $\Gamma = \{\chi_1, \dots, \chi_n\}$

$\Delta_1 = \{x_1 : \chi_1, \dots, x_n : \chi_n\}$ and $\Delta_2 = \{y_1 : \chi_1, \dots, y_n : \chi_n\}$

let t_2' be obtained from t_2 by replacing every y_i with x_i

$\Delta_1 \vdash t_2' : \psi$

Theorem (Curry–Howard)

2. if $\Gamma \vdash_{\mathsf{h}} \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\mathsf{types}(\Delta) = \Gamma$

Proof

induction on derivation of $\Gamma \vdash_{\mathsf{h}} \varphi$

interesting case: φ is obtained by modus ponens

$\Gamma \vdash_{\mathsf{h}} \psi \to \varphi$ and $\Gamma \vdash_{\mathsf{h}} \psi$

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for some t_1, Δ_1, t_2, Δ_2 with $\mathsf{types}(\Delta_1) = \mathsf{types}(\Delta_2) = \Gamma$

suppose $\Gamma = \{\chi_1, \dots, \chi_n\}$

$\Delta_1 = \{x_1 : \chi_1, \dots, x_n : \chi_n\}$ and $\Delta_2 = \{y_1 : \chi_1, \dots, y_n : \chi_n\}$

let t_2' be obtained from t_2 by replacing every y_i with x_i

$\Delta_1 \vdash t_2' : \psi$ and thus $\Delta_1 \vdash t_1\, t_2' : \varphi$

Outline

1. Summary of Previous Lecture
2. Evaluation
3. Hilbert Systems
4. Curry–Howard Isomorphism
5. **Intuitionistic Propositional Logic**
6. Summary

Definition

Hilbert system for intuitionistic propositional logic consists of modus ponens and axioms

① $\varphi \to \psi \to \varphi$

② $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$

Definition

Hilbert system for intuitionistic propositional logic consists of modus ponens and axioms

① $\varphi \to \psi \to \varphi$

② $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$

③ $\varphi \land \psi \to \varphi$

④ $\varphi \land \psi \to \psi$

⑤ $\varphi \to \psi \to \varphi \land \psi$

Definition

Hilbert system for intuitionistic propositional logic consists of modus ponens and axioms

1. $\varphi \to \psi \to \varphi$
2. $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
3. $\varphi \wedge \psi \to \varphi$
4. $\varphi \wedge \psi \to \psi$
5. $\varphi \to \psi \to \varphi \wedge \psi$
6. $\varphi \to \varphi \vee \psi$
7. $\psi \to \varphi \vee \psi$
8. $(\varphi \to \chi) \to (\psi \to \chi) \to \varphi \vee \psi \to \chi$

Definition

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⑤ $\varphi \to \psi \to \varphi \wedge \psi$

⑥ $\varphi \to \varphi \vee \psi$

⑦ $\psi \to \varphi \vee \psi$

⑧ $(\varphi \to \chi) \to (\psi \to \chi) \to \varphi \vee \psi \to \chi$

⑨ $\bot \to \varphi$

Definition

Hilbert system for intuitionistic propositional logic consists of modus ponens and axioms

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3. $\varphi \land \psi \to \varphi$
4. $\varphi \land \psi \to \psi$
5. $\varphi \to \psi \to \varphi \land \psi$
6. $\varphi \to \varphi \lor \psi$
7. $\psi \to \varphi \lor \psi$
8. $(\varphi \to \chi) \to (\psi \to \chi) \to \varphi \lor \psi \to \chi$
9. $\bot \to \varphi$

Remarks

- $\neg\varphi$ is shortcut for $\varphi \to \bot$

Definition

Hilbert system for intuitionistic propositional logic consists of modus ponens and axioms

① $\varphi \to \psi \to \varphi$

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④ $\varphi \wedge \psi \to \psi$

⑤ $\varphi \to \psi \to \varphi \wedge \psi$

⑥ $\varphi \to \varphi \vee \psi$

⑦ $\psi \to \varphi \vee \psi$

⑧ $(\varphi \to \chi) \to (\psi \to \chi) \to \varphi \vee \psi \to \chi$

⑨ $\bot \to \varphi$

Remarks

- $\neg\varphi$ is shortcut for $\varphi \to \bot$
- adding axiom $\varphi \vee \neg\varphi$ (law of excluded middle) gives (classical) propositional logic

Definition

Hilbert system for intuitionistic propositional logic consists of modus ponens and axioms

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⑨ $\bot \to \varphi$

Remarks

- $\neg \varphi$ is shortcut for $\varphi \to \bot$
- adding axiom $\varphi \vee \neg \varphi$ (law of excluded middle) gives (classical) propositional logic
- intuitionistic propositional logic is known as IPC in literature

Theorem

Hilbert system for IPC is sound and complete with respect to Kripke models:

$$\Gamma \vdash_{\mathsf{h}} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash \varphi$$

Theorem

Hilbert system for IPC is sound and complete with respect to Kripke models:

$$\Gamma \vdash_{\mathsf{h}} \varphi \quad \iff \quad \Gamma \Vdash \varphi$$

Theorem (Finite Model Property)

$\vdash_{\mathsf{h}} \varphi \quad \iff \quad \mathcal{C} \Vdash \varphi$ for all finite Kripke models $\mathcal{C}$

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Theorem

problem

- instance: formula φ
- question: $\vdash_{\mathsf{h}} \varphi$?

is decidable

Theorem

Hilbert system for IPC is sound and complete with respect to Kripke models:

$$\Gamma \vdash_{\mathsf{h}} \varphi \iff \Gamma \Vdash \varphi$$

Theorem (Finite Model Property)

$\vdash_{\mathsf{h}} \varphi \iff \mathcal{C} \Vdash \varphi$ for all finite Kripke models $\mathcal{C}$

Theorem

problem

instance: formula φ

question: $\vdash_{\mathsf{h}} \varphi$?

is decidable and PSPACE–complete

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_{\mathsf{h}} \neg\neg\varphi$

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Remark

Glivenko's theorem does not extend to predicate logic

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_h \neg\neg\varphi$

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Definition (Gödel's Negative Translation)

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_{\mathsf{h}} \neg\neg\varphi$

Remark

Glivenko's theorem does not extend to predicate logic

Definition (Gödel's Negative Translation)

- $p^{\mathsf{n}} = \neg\neg p$ for propositional atoms p

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_{\mathsf{h}} \neg\neg\varphi$

Remark

Glivenko's theorem does not extend to predicate logic

Definition (Gödel's Negative Translation)

- $p^{\mathsf{n}} = \neg\neg p$ for propositional atoms p
- $(\varphi \wedge \psi)^{\mathsf{n}} = \varphi^{\mathsf{n}} \wedge \psi^{\mathsf{n}}$

Theorem (Glivenko 1929)

$\vdash \varphi \quad \Longleftrightarrow \quad \vdash_{\mathsf{h}} \neg\neg\varphi$

Remark

Glivenko's theorem does not extend to predicate logic

Definition (Gödel's Negative Translation)

- $p^{\mathsf{n}} = \neg\neg p$ for propositional atoms p
- $(\varphi \wedge \psi)^{\mathsf{n}} = \varphi^{\mathsf{n}} \wedge \psi^{\mathsf{n}}$
- $(\varphi \vee \psi)^{\mathsf{n}} = \neg(\neg\varphi^{\mathsf{n}} \wedge \neg\psi^{\mathsf{n}})$

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_{\mathsf{h}} \neg\neg\varphi$

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- $(\varphi \to \psi)^{\mathsf{n}} = \varphi^{\mathsf{n}} \to \psi^{\mathsf{n}}$

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_h \neg\neg\varphi$

Remark

Glivenko's theorem does not extend to predicate logic

Definition (Gödel's Negative Translation)

- $p^n = \neg\neg p$ for propositional atoms p
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$
- $(\varphi \vee \psi)^n = \neg(\neg\varphi^n \wedge \neg\psi^n)$
- $(\varphi \to \psi)^n = \varphi^n \to \psi^n$
- $\bot^n = \bot$

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_{\mathsf{h}} \neg\neg\varphi$

Remark

Glivenko's theorem does not extend to predicate logic

Definition (Gödel's Negative Translation)

- $p^{\mathsf{n}} = \neg\neg p$ for propositional atoms p
- $(\varphi \wedge \psi)^{\mathsf{n}} = \varphi^{\mathsf{n}} \wedge \psi^{\mathsf{n}}$
- $(\varphi \vee \psi)^{\mathsf{n}} = \neg(\neg\varphi^{\mathsf{n}} \wedge \neg\psi^{\mathsf{n}})$
- $(\varphi \to \psi)^{\mathsf{n}} = \varphi^{\mathsf{n}} \to \psi^{\mathsf{n}}$
- $\bot^{\mathsf{n}} = \bot$

Theorem

$\vdash \varphi \iff \vdash_{\mathsf{h}} \varphi^{\mathsf{n}}$

Outline

Important Concepts

- φ^{n}
- $\vdash_{\mathsf{h}}$
- Curry–Howard isomorphism
- finite model property
- Glivenko's theorem
- Gödel's negative translation
- Hilbert system
- intuitionistic propositional logic

Important Concepts

- φ^{n}
- $\vdash_{h}$
- Curry–Howard isomorphism
- finite model property
- Glivenko's theorem
- Gödel's negative translation
- Hilbert system
- intuitionistic propositional logic

homework for January 15