universität innsbruck

WS 2023 lecture 13

Computability Theory

Aart Middeldorp

Outline

1. **Summary of Previous Lecture**
2. **β–Reduction**
3. **Church–Rosser Theorem**
4. **λ–Definability**
5. **η–Reduction**
6. **Normalization Theorem**
7. **Test Practice**
8. **Summary**

Definition

Hilbert system (for implication fragment) consists of two axioms and modus ponens:

$$\frac{}{\varphi \to \psi \to \varphi} \qquad \frac{}{(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi} \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Deduction Theorem

$$\Gamma \cup \{\varphi\} \vdash_{\mathsf{h}} \psi \quad \iff \quad \Gamma \vdash_{\mathsf{h}} \varphi \to \psi$$

Theorem

Hilbert system is sound and complete with respect to Kripke models for implication fragment:

$$\Gamma \vdash_{\mathsf{h}} \varphi \quad \iff \quad \Gamma \Vdash \varphi$$

Theorem (Curry–Howard)

1. if $\Gamma \vdash t : \tau$ then $\text{types}(\Gamma) \vdash_h \tau$
2. if $\Gamma \vdash_h \varphi$ then $\Delta \vdash t : \varphi$ for some t and Δ with $\text{types}(\Delta) = \Gamma$

Definition

Hilbert system for intuitionistic propositional logic consists of modus ponens and axioms

① $\varphi \to \psi \to \varphi$

② $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$

③ $\varphi \vee \psi \to \varphi$

④ $\varphi \vee \psi \to \psi$

⑤ $\varphi \to \psi \to \varphi \wedge \psi$

⑥ $\varphi \to \varphi \vee \psi$

⑦ $\psi \to \varphi \vee \psi$

⑧ $(\varphi \to \chi) \to (\psi \to \chi) \to \varphi \vee \psi \to \chi$

⑨ $\bot \to \varphi$

Remarks

- $\neg\varphi$ is shortcut for $\varphi \to \bot$
- adding axiom $\varphi \vee \neg\varphi$ (law of excluded middle) gives (classical) propositional logic

Theorem

Hilbert system is sound and complete with respect to Kripke models:

$$\Gamma \vdash_{\mathsf{h}} \varphi \iff \Gamma \Vdash \varphi$$

Theorem (Finite Model Property)

$\vdash_{\mathsf{h}} \varphi \iff \mathcal{C} \Vdash \varphi$ for all finite Kripke models $\mathcal{C}$

Theorem (Glivenko 1929)

$\vdash \varphi \iff \vdash_{\mathsf{h}} \neg\neg\varphi$

Theorem

problem

instance: formula φ

question: $\vdash_{\mathsf{h}} \varphi$?

is decidable and PSPACE–complete

Definition (Gödel's Negative Translation)

- $p^{\mathsf{n}} = \neg\neg p$ for propositional atoms p
- $(\varphi \wedge \psi)^{\mathsf{n}} = \varphi^{\mathsf{n}} \wedge \psi^{\mathsf{n}}$
- $(\varphi \vee \psi)^{\mathsf{n}} = \neg(\neg\varphi^{\mathsf{n}} \wedge \neg\psi^{\mathsf{n}})$
- $(\varphi \to \psi)^{\mathsf{n}} = \varphi^{\mathsf{n}} \to \psi^{\mathsf{n}}$
- $\bot^{\mathsf{n}} = \bot$

Theorem

$\vdash \varphi \iff \vdash_{\mathsf{h}} \varphi^{\mathsf{n}}$

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course−of−values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s−m−n theorem, total recursive functions, undecidability, while programs, ...

Part II: Combinatory Logic and Lambda Calculus

α−equivalence, abstraction, arithmetization, β−reduction, CL−representability, combinators, combinatorial completeness, Church numerals, Church−Rosser theorem, Curry−Howard isomorphism, de Bruijn notation, η−reduction, fixed point theorem, intuitionistic propositional logic, λ−definability, normalization theorem, termination, typing, undecidability, Z property, ...

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Literature (Combinatory Logic and Lambda Calculus)

- Henk Barendregt
  The Lambda Calculus, Its Syntax and Semantics
  North Holland, 1984
- Henk Barendregt, Wil Dekkers and Richard Statman
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  Cambridge University Press, 2014
- Chris Hankin
  An Introduction to Lambda Calculi for Computer Scientists
  King's College Publications, 2000
- J. Roger Hindley and Jonathan P. Seldin
  Lambda−Calculus and Combinators, an Introduction
  Cambridge University Press, 2008

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Examples

$(\lambda x.x) \qquad ((\lambda x.(xx))(\lambda y.(yy))) \qquad (\lambda f.(\lambda x.(f(fx))))$

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Backus–Naur Form

$$M, N \ ::= \ x \mid (MN) \mid (\lambda x.M)$$

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Terminology

$$\lambda x.M$$

- λx is binder

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Terminology

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- M is scope of binder λx
- occurrence of x in $\lambda x.M$ is bound

Notation

$M \equiv N$ if M and N are identical

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$$\mathsf{FV}(x) = \{x\}$$
$$\mathsf{FV}(MN) = \mathsf{FV}(M) \cup \mathsf{FV}(N)$$
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$$M \equiv (\lambda x.xy)(\lambda y.yz)$$

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Example

$M \equiv (\lambda x.xy)(\lambda y.yz)$ $\mathsf{FV}(M) = \{y, z\}$

Definition (Renaming)

$$x\{y/x\} \equiv y$$

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α–equivalence is smallest congruence relation $\equiv_\alpha$ on lambda terms such that

$$\lambda x.M \equiv_\alpha \lambda y.(M\{y/x\})$$

for all terms M and variables y that do not occur in M

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α–equivalence

$$\frac{y \notin M}{\lambda x.M \equiv_\alpha \lambda y.(M\{y/x\})} \quad (\alpha)$$

α–equivalence

(reflexivity) $$\frac{}{M \equiv_\alpha M}$$

(symmetry) $$\frac{M \equiv_\alpha N}{N \equiv_\alpha M}$$

(transitivity) $$\frac{M \equiv_\alpha N \quad N \equiv_\alpha P}{M \equiv_\alpha P}$$

$$\frac{y \notin M}{\lambda x.M \equiv_\alpha \lambda y.(M\{y/x\})} \ (\alpha)$$

α–equivalence

$$\text{(reflexivity)} \quad \frac{}{M \equiv_\alpha M} \qquad \frac{M \equiv_\alpha M' \quad N \equiv_\alpha N'}{MN \equiv_\alpha M'N'} \quad \text{(congruence)}$$

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Examples

$$\lambda x.y \equiv_\alpha \lambda z.y$$

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Examples

$\lambda x.y \equiv_\alpha \lambda z.y \qquad \lambda x.y \not\equiv_\alpha \lambda y.y \qquad (\lambda x.y)z \not\equiv_\alpha (\lambda x.w)z$

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$$\lambda x.y \equiv_\alpha \lambda z.y \qquad \lambda x.y \not\equiv_\alpha \lambda y.y \qquad (\lambda x.y)z \not\equiv_\alpha (\lambda x.w)z \qquad (\lambda x.y)(\lambda z.z) \equiv_\alpha (\lambda z.y)(\lambda z.z)$$

Barendregt's Variable Convention

free variables are different from bound variables in lambda terms occurring in certain context

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Convention

lambda terms are identified up to α–equivalence

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one-step β–reduction is smallest relation $\to_\beta$ on lambda terms satisfying

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$$\begin{aligned}(\lambda x.y)((\lambda z.zz)(\lambda w.w)) &\rightarrow_\beta (\lambda x.y)((\lambda w.w)(\lambda w.w)) \\ &\rightarrow_\beta (\lambda x.y)(\lambda w.w) \rightarrow_\beta y\end{aligned}$$

Definitions

- β–**normal form** is lambda term without β–redexes

Definition

one-step β–reduction is smallest relation $\rightarrow_\beta$ on lambda terms satisfying

$$(\beta) \quad \frac{}{(\lambda x.M)N \rightarrow_\beta M[N/x]} \qquad \frac{M \rightarrow_\beta M'}{MN \rightarrow_\beta M'N} \quad \text{(congruence)}$$

$$(\xi) \quad \frac{M \rightarrow_\beta M'}{\lambda x.M \rightarrow_\beta \lambda x.M'} \qquad \frac{N \rightarrow_\beta N'}{MN \rightarrow_\beta MN'} \quad \text{(congruence)}$$

Example

$$\begin{aligned}(\lambda x.y)((\lambda z.zz)(\lambda w.w)) &\rightarrow_\beta (\lambda x.y)((\lambda w.w)(\lambda w.w)) \\ &\rightarrow_\beta (\lambda x.y)(\lambda w.w) \rightarrow_\beta y\end{aligned}$$

Definitions

- β–normal form is lambda term without β–redexes
- β–conversion ($=_\beta$) is transitive symmetric reflexive closure of $\rightarrow_\beta$

Definition

lambda term N is fixed point of lambda term F if $FN =_\beta N$

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Definition (Turing's Fixed Point Combinator)

$\Theta \equiv AA$ with $A = \lambda xy.y(xxy)$

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Proof

$N \equiv \Theta F$ is fixed point of F:

$$N \equiv (\lambda xy.y(xxy))AF$$

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Theorem

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Proof

$N \equiv \Theta F$ is fixed point of F:

$$N \equiv (\lambda xy.y(xxy))AF \to_\beta (\lambda y.y(AAy))F$$

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$N \equiv \Theta F$ is fixed point of F:

$$N \equiv (\lambda xy.y(xxy))AF \to_\beta (\lambda y.y(AAy))F \to_\beta F(AAF) \equiv F(\Theta F)$$

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Theorem

every lambda term has fixed point

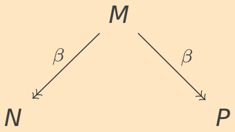
Proof

$N \equiv \Theta F$ is fixed point of F:

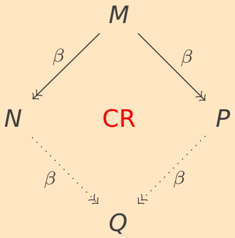
$$N \equiv (\lambda xy.y(xxy))AF \to_\beta (\lambda y.y(AAy))F \to_\beta F(AAF) \equiv F(\Theta F) \equiv FN$$

Outline

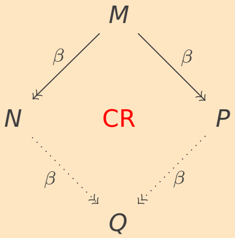
Church–Rosser Theorem



Church–Rosser Theorem



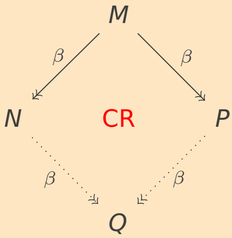
Church–Rosser Theorem



Corollary

- $M =_\beta N \quad \Longrightarrow \quad \exists\, Q$ such that $M \to_\beta^* Q$ and $N \to_\beta^* Q$

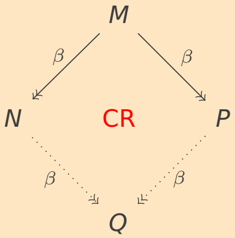
Church–Rosser Theorem



Corollary

- $M =_\beta N \quad \Longrightarrow \quad \exists\, Q$ such that $M \to_\beta^* Q$ and $N \to_\beta^* Q$
- $M =_\beta N$ and N is β–normal form $\quad \Longrightarrow \quad M \to_\beta^* N$

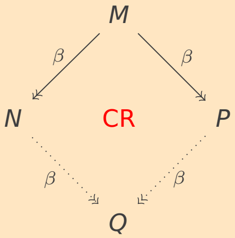
Church–Rosser Theorem



Corollary

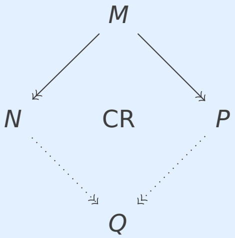
- $M =_\beta N \implies \exists\, Q$ such that $M \to_\beta^* Q$ and $N \to_\beta^* Q$
- $M =_\beta N$ and N is β–normal form $\implies M \to_\beta^* N$
- $M =_\beta N$ and M, N are β–normal forms $\implies M \equiv_\alpha N$

Church–Rosser Theorem

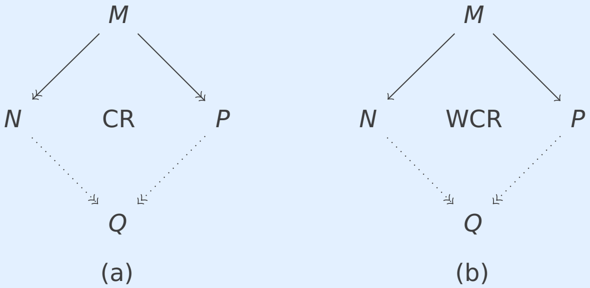


Corollary

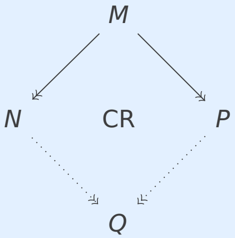
- $M =_\beta N \quad \Longrightarrow \quad \exists\, Q$ such that $M \to_\beta^* Q$ and $N \to_\beta^* Q$
- $M =_\beta N$ and N is β–normal form $\quad \Longrightarrow \quad M \to_\beta^* N$
- $M =_\beta N$ and M, N are β–normal forms $\quad \Longrightarrow \quad M \equiv_\alpha N$
- $M =_\beta N \quad \Longrightarrow \quad$ both or neither of M, N have β–normal form



(a)



- β–reduction satisfies (b)



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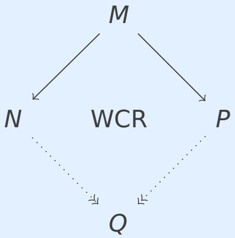
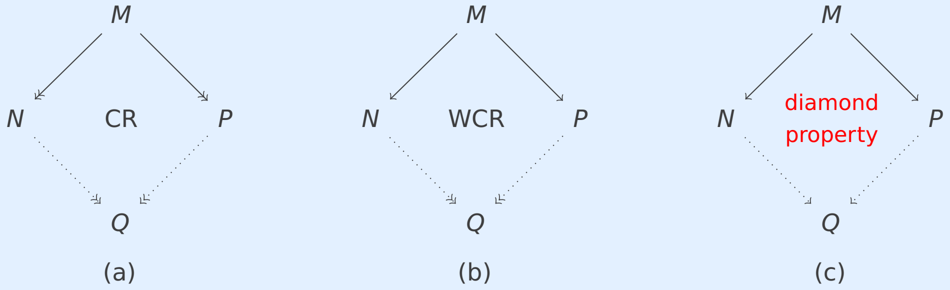
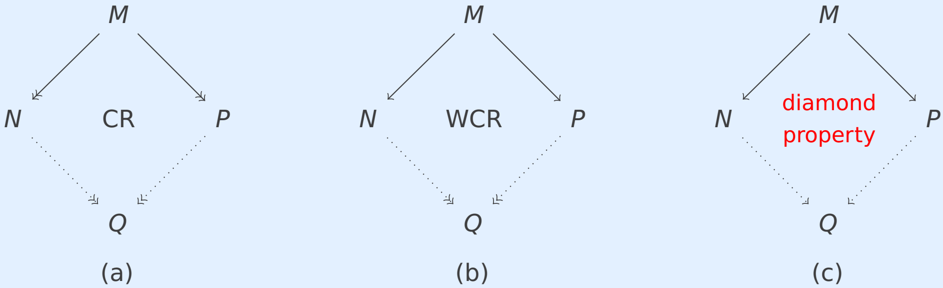


(b)

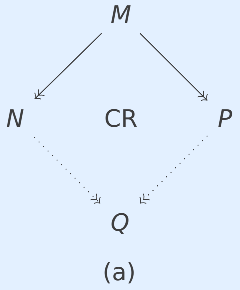
- β–reduction satisfies (b)
- (b) $\not\Longrightarrow$ (a)



- β–reduction satisfies (b)
- (b) $\;\not\Longrightarrow\;$ (a)
- (c) $\;\Longrightarrow\;$ (a)



- β–reduction satisfies (b)
- (b) $\not\Longrightarrow$ (a)
- (c) $\Longrightarrow$ (a)
- β–reduction does not satisfy (c)



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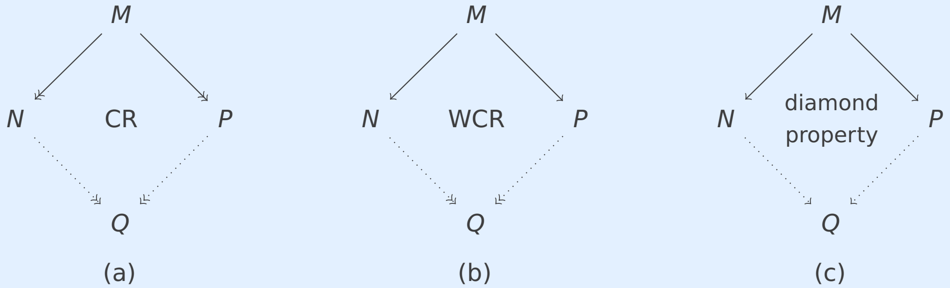
- β–reduction does not satisfy (c): $\qquad (\lambda x.xx)((\lambda y.y)z)$



- β–reduction satisfies (b)
- (b) $\not\Longrightarrow$ (a)
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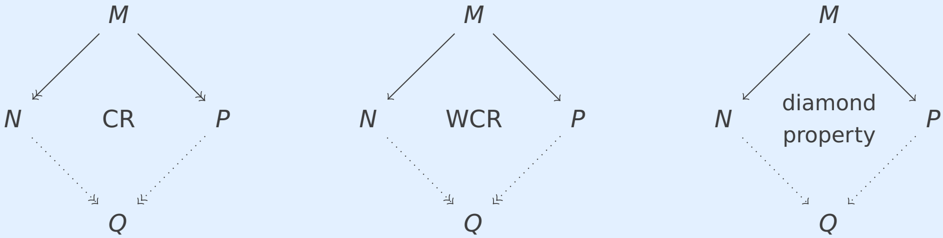

(a) (b) (c)

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- β–reduction does not satisfy (c): $(\lambda x.xx)z \; {}_{\beta}\!\leftarrow (\lambda x.xx)((\lambda y.y)z) \rightarrow_{\beta} (\lambda y.y)z((\lambda y.y)z)$

Definition (Parallel Reduction)

$$\frac{}{M \;\Longrightarrow\!\!\!\!\!\!\!\!\;\;\|\;_\beta\; M}$$

Definition (Parallel Reduction)

$$\frac{}{M \;\twoheadrightarrow\!\!\!\!\!\!\parallel\;_\beta\; M} \qquad \frac{}{(\lambda x.M)N \;\twoheadrightarrow\!\!\!\!\!\!\parallel\;_\beta\; M[N/x]}$$

Definition (Parallel Reduction)

$$\frac{}{M \;\twoheadrightarrow\!\!\!\!\!\!\!\!\;\,+\!\!\!\!\;\; _\beta\; M}$$

$$\frac{}{M \mathrel{\Vert\!\!\!\rightarrow}_\beta M} \qquad \frac{}{(\lambda x.M)N \mathrel{\Vert\!\!\!\rightarrow}_\beta M[N/x]} \qquad \frac{M \mathrel{\Vert\!\!\!\rightarrow}_\beta M'}{\lambda x.M \mathrel{\Vert\!\!\!\rightarrow}_\beta \lambda x.M'}$$

Definition (Parallel Reduction)

$$\frac{}{M \mathrel{\rightarrow\!\!\!\!\!\!\!\parallel\;}_\beta M} \qquad \frac{}{(\lambda x.M)N \mathrel{\rightarrow\!\!\!\!\!\!\!\parallel\;}_\beta M[N/x]} \qquad \frac{M \mathrel{\rightarrow\!\!\!\!\!\!\!\parallel\;}_\beta M'}{\lambda x.M \mathrel{\rightarrow\!\!\!\!\!\!\!\parallel\;}_\beta \lambda x.M'} \qquad \frac{M \mathrel{\rightarrow\!\!\!\!\!\!\!\parallel\;}_\beta M' \qquad N \mathrel{\rightarrow\!\!\!\!\!\!\!\parallel\;}_\beta N'}{MN \mathrel{\rightarrow\!\!\!\!\!\!\!\parallel\;}_\beta M'N'}$$

Definition (Parallel Reduction)

$$\frac{}{M \twoheadrightarrow\!\!\!\!\!\!\!\mid\mid_\beta M} \qquad \frac{}{(\lambda x.M)N \twoheadrightarrow\!\!\!\!\!\!\!\mid\mid_\beta M[N/x]} \qquad \frac{M \twoheadrightarrow\!\!\!\!\!\!\!\mid\mid_\beta M'}{\lambda x.M \twoheadrightarrow\!\!\!\!\!\!\!\mid\mid_\beta \lambda x.M'} \qquad \frac{M \twoheadrightarrow\!\!\!\!\!\!\!\mid\mid_\beta M' \quad N \twoheadrightarrow\!\!\!\!\!\!\!\mid\mid_\beta N'}{MN \twoheadrightarrow\!\!\!\!\!\!\!\mid\mid_\beta M'N'}$$

Problem

$\twoheadrightarrow\!\!\!\!\!\!\!\mid\mid$ lacks diamond property

Definition (Parallel Reduction)

$$\frac{}{M \twoheadrightarrow_\beta M} \qquad \frac{}{(\lambda x.M)N \twoheadrightarrow_\beta M[N/x]} \qquad \frac{M \twoheadrightarrow_\beta M'}{\lambda x.M \twoheadrightarrow_\beta \lambda x.M'} \qquad \frac{M \twoheadrightarrow_\beta M' \quad N \twoheadrightarrow_\beta N'}{MN \twoheadrightarrow_\beta M'N'}$$

Problem

$\twoheadrightarrow$ lacks diamond property: $(\lambda x.x)I \ {}_\beta\!\twoheadleftarrow (\lambda x.(\lambda y.x)I)(II) \twoheadrightarrow_\beta (\lambda y.II)I$ with $I = \lambda x.x$

Definition (Parallel Reduction)

$$\frac{}{M \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta M} \qquad \frac{}{(\lambda x.M)N \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta M[N/x]} \qquad \frac{M \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta M'}{\lambda x.M \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta \lambda x.M'} \qquad \frac{M \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta M' \quad N \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta N'}{MN \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta M'N'}$$

Problem

$\mathrel{+\!\!\!\!\!\!\!\rightarrow}$ lacks diamond property: $(\lambda x.x)I \;{}_\beta\!\!\mathrel{\leftarrow\!\!\!\!\!\!\!+} (\lambda x.(\lambda y.x)I)(II) \mathrel{+\!\!\!\!\!\!\!\rightarrow}_\beta (\lambda y.II)I$ with $I = \lambda x.x$

Definition (Parallel Reduction Revisited)

$$\frac{}{M \mathrel{\multimap\!\!\!\rightarrow}_\beta M} \qquad \frac{M \mathrel{\multimap\!\!\!\rightarrow}_\beta M' \quad N \mathrel{\multimap\!\!\!\rightarrow}_\beta N'}{(\lambda x.M)N \mathrel{\multimap\!\!\!\rightarrow}_\beta M'[N'/x]} \qquad \frac{M \mathrel{\multimap\!\!\!\rightarrow}_\beta M'}{\lambda x.M \mathrel{\multimap\!\!\!\rightarrow}_\beta \lambda x.M'} \qquad \frac{M \mathrel{\multimap\!\!\!\rightarrow}_\beta M' \quad N \mathrel{\multimap\!\!\!\rightarrow}_\beta N'}{MN \mathrel{\multimap\!\!\!\rightarrow}_\beta M'N'}$$

Definition (Parallel Reduction)

$$\frac{}{M \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta M} \qquad \frac{}{(\lambda x.M)N \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta M[N/x]} \qquad \frac{M \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta M'}{\lambda x.M \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta \lambda x.M'} \qquad \frac{M \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta M' \qquad N \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta N'}{MN \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta M'N'}$$

Problem

$\mathrel{+\!\!\!\!\!\!\!\longrightarrow}$ lacks diamond property: $(\lambda x.x)I \;{}_\beta\!\mathrel{\longleftarrow\!\!\!\!\!\!\!+} (\lambda x.(\lambda y.x)I)(II) \mathrel{+\!\!\!\!\!\!\!\longrightarrow}_\beta (\lambda y.II)I$ with $I = \lambda x.x$

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$$\frac{}{M \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta M} \qquad \frac{M \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta M' \qquad N \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta N'}{(\lambda x.M)N \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta M'[N'/x]} \qquad \frac{M \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta M'}{\lambda x.M \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta \lambda x.M'} \qquad \frac{M \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta M' \qquad N \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta N'}{MN \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta M'N'}$$

Lemma

$\to_\beta \;\subseteq\; \mathrel{\circ\!\!\!\!\!\!\longrightarrow}_\beta \;\subseteq\; \to_\beta^*$

Lemma (Substitution)

if $M \multimap\!\!\rightarrow_\beta M'$ and $U \multimap\!\!\rightarrow_\beta U'$ then $M[U/y] \multimap\!\!\rightarrow_\beta M'[U'/y]$

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Definition

M^* is maximal parallel one-step reduct of M:

① $x^* = x$

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② $(PN)^* = P^*N^*$ if PN is no β–redex

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③ $((\lambda x.Q)N)^* = Q^*[N^*/x]$

④ $(\lambda x.N)^* = \lambda x.N^*$

Lemma

if $M \multimap\!\!\!\rightarrow_\beta N$ then $N \multimap\!\!\!\rightarrow_\beta M^*$

Lemma (Diamond Property of Parallel Reduction)

$\forall\, M, N, P \in \Lambda$ such that $M \twoheadrightarrow\!\!\!\!\!\circ\;_\beta N$ and $M \twoheadrightarrow\!\!\!\!\!\circ\;_\beta P$

$\exists\, Q \in \Lambda$ such that $N \twoheadrightarrow\!\!\!\!\!\circ\;_\beta Q$ and $P \twoheadrightarrow\!\!\!\!\!\circ\;_\beta Q$

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Proof

take $Q \equiv M^*$

Lemma (Diamond Property of Parallel Reduction)

$\forall\, M, N, P \in \Lambda$ such that $M \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta P$

$\exists\, Q \in \Lambda$ such that $N \twoheadrightarrow_\beta Q$ and $P \twoheadrightarrow_\beta Q$

Proof

take $Q \equiv M^*$

Corollary

β–reduction has Church–Rosser property

Outline

Definitions

- $\mathsf{T} \equiv \lambda xy.x$

Definitions

- $\mathsf{T} \equiv \lambda xy.x \qquad \mathsf{F} \equiv \lambda xy.y$

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- $\mathsf{T} \equiv \lambda xy.x \qquad \mathsf{F} \equiv \lambda xy.y \qquad \mathsf{and} \equiv \lambda ab.ab\,\mathsf{F}$

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Lemmata

- $\mathsf{and}\,\mathsf{T}\,\mathsf{T} \to_\beta^* \mathsf{T} \qquad \mathsf{and}\,\mathsf{T}\,\mathsf{F} \to_\beta^* \mathsf{F} \qquad \mathsf{and}\,\mathsf{F}\,\mathsf{T} \to_\beta^* \mathsf{F} \qquad \mathsf{and}\,\mathsf{F}\,\mathsf{F} \to_\beta^* \mathsf{F}$

Definitions

- $\mathsf{T} \equiv \lambda xy.x \qquad \mathsf{F} \equiv \lambda xy.y \qquad \mathsf{and} \equiv \lambda ab.ab\,\mathsf{F}$
- $\mathsf{ite} \equiv \lambda x.x$

Lemmata

- $\mathsf{and}\,\mathsf{T}\,\mathsf{T} \to_\beta^* \mathsf{T} \qquad \mathsf{and}\,\mathsf{T}\,\mathsf{F} \to_\beta^* \mathsf{F} \qquad \mathsf{and}\,\mathsf{F}\,\mathsf{T} \to_\beta^* \mathsf{F} \qquad \mathsf{and}\,\mathsf{F}\,\mathsf{F} \to_\beta^* \mathsf{F}$

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Definitions

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- $\mathsf{ite}\,\mathsf{T}\,M\,N \to_\beta^* M \qquad \mathsf{ite}\,\mathsf{F}\,M\,N \to_\beta^* N$

Definitions (Church Numerals)

- for every natural number n

$$\underline{n} \equiv \lambda fx.f^n x \quad \text{where} \quad F^n M \equiv \begin{cases} M & \text{if } n = 0 \\ F(F^{n-1}M) & \text{if } n > 0 \end{cases}$$

Definitions

- $\mathsf{T} \equiv \lambda xy.x \qquad \mathsf{F} \equiv \lambda xy.y \qquad \mathsf{and} \equiv \lambda ab.ab\,\mathsf{F}$
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Lemmata

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- $\mathsf{succ} \equiv \lambda nfx.f(nfx)$

Lemma

$\mathsf{succ}\ \underline{n} \rightarrow_\beta^* \underline{n+1}$

Lemma

$\mathsf{succ}\ \underline{n} \to_\beta^* \underline{n+1}$

Proof

$$\mathsf{succ}\ \underline{n} \equiv (\lambda n f x. f(n f x))(\lambda f x. f^n x)$$

Lemma

$\text{succ}\ \underline{n} \rightarrow_\beta^* \underline{n+1}$

Proof

$$\text{succ}\ \underline{n} \equiv (\lambda n f x. f(n f x))(\lambda f x. f^n x) \rightarrow_\beta \lambda f x. f((\lambda f x. f^n x) f x)$$

Lemma

$\text{succ}\ \underline{n} \to_\beta^* \underline{n+1}$

Proof

$$\text{succ}\ \underline{n} \equiv (\lambda n f x. f(n f x))(\lambda f x. f^n x) \to_\beta \lambda f x. f((\lambda f x. f^n x) f x) \to_\beta \lambda f x. f((\lambda x. f^n x) x)$$

Lemma

$\text{succ}\ \underline{n} \to_\beta^* \underline{n+1}$

Proof

$$\text{succ}\ \underline{n} \equiv (\lambda n f x. f\,(n\,f\,x))(\lambda f x. f^n x) \to_\beta \lambda f x. f\,((\lambda f x. f^n x)\,f\,x) \to_\beta \lambda f x. f\,((\lambda x. f^n x)\,x)$$
$$\to_\beta \lambda f x. f\,(f^n x)$$

Lemma

$\text{succ}\ \underline{n} \rightarrow_\beta^* \underline{n+1}$

Proof

$$\begin{aligned}\text{succ}\ \underline{n} \equiv (\lambda n f x. f(n f x))(\lambda f x. f^n x) &\rightarrow_\beta \lambda f x. f((\lambda f x. f^n x) f x) \rightarrow_\beta \lambda f x. f((\lambda x. f^n x) x) \\ &\rightarrow_\beta \lambda f x. f(f^n x) \equiv \lambda f x. f^{n+1} x \equiv \underline{n+1}\end{aligned}$$

Lemma

$\text{succ}\ \underline{n} \to_\beta^* \underline{n+1}$

Proof

$$\text{succ}\ \underline{n} \equiv (\lambda n f x. f(n f x))(\lambda f x. f^n x) \to_\beta \lambda f x. f((\lambda f x. f^n x) f x) \to_\beta \lambda f x. f((\lambda x. f^n x) x)$$
$$\to_\beta \lambda f x. f(f^n x) \equiv \lambda f x. f^{n+1} x \equiv \underline{n+1}$$

Definitions

- $\text{zero?} \equiv \lambda n. n(\lambda x. \text{F})\,\text{T}$

Lemma

$\text{succ}\,\underline{n} \to_\beta^* \underline{n+1}$

Proof

$$\text{succ}\,\underline{n} \equiv (\lambda n f x.f(n f x))(\lambda f x.f^n x) \to_\beta \lambda f x.f((\lambda f x.f^n x) f x) \to_\beta \lambda f x.f((\lambda x.f^n x) x)$$
$$\to_\beta \lambda f x.f(f^n x) \equiv \lambda f x.f^{n+1} x \equiv \underline{n+1}$$

Definitions

- $\text{zero?} \equiv \lambda n.n(\lambda x.\text{F})\,\text{T}$

Lemmata

- $\text{zero?}\,\underline{n} \to_\beta^* \begin{cases} \text{T} & \text{if } n = 0 \\ \text{F} & \text{if } n > 0 \end{cases}$

Lemma

$\text{succ}\ \underline{n} \rightarrow_\beta^* \underline{n+1}$

Proof

$$\begin{aligned} \text{succ}\ \underline{n} \equiv (\lambda nfx.f(n\,f\,x))(\lambda fx.f^n x) &\rightarrow_\beta \lambda fx.f((\lambda fx.f^n x)\,f\,x) \rightarrow_\beta \lambda fx.f((\lambda x.f^n x)\,x) \\ &\rightarrow_\beta \lambda fx.f(f^n x) \equiv \lambda fx.f^{n+1} x \equiv \underline{n+1} \end{aligned}$$

Definitions

- $\text{zero?} \equiv \lambda n.n(\lambda x.\text{F})\,\text{T}$ $\qquad$ $\text{add} \equiv \lambda nmfx.nf(mfx)$

Lemmata

- $\text{zero?}\ \underline{n} \rightarrow_\beta^* \begin{cases} \text{T} & \text{if } n = 0 \\ \text{F} & \text{if } n > 0 \end{cases}$

Lemma

$\mathsf{succ}\ \underline{n} \to_\beta^* \underline{n+1}$

Proof

$$\mathsf{succ}\ \underline{n} \equiv (\lambda nfx.f(nfx))(\lambda fx.f^n x) \to_\beta \lambda fx.f((\lambda fx.f^n x)fx) \to_\beta \lambda fx.f((\lambda x.f^n x)x)$$
$$\to_\beta \lambda fx.f(f^n x) \equiv \lambda fx.f^{n+1}x \equiv \underline{n+1}$$

Definitions

- $\mathsf{zero?} \equiv \lambda n.n(\lambda x.\mathsf{F})\mathsf{T}$ $\quad$ $\mathsf{add} \equiv \lambda nmfx.nf(mfx)$ $\quad$ $\mathsf{mul} \equiv \lambda nmf.n(mf)$

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- $\mathsf{zero?}\ \underline{n} \to_\beta^* \begin{cases} \mathsf{T} & \text{if } n = 0 \\ \mathsf{F} & \text{if } n > 0 \end{cases}$

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$$\text{succ}\ \underline{n} \equiv (\lambda nfx.f(nfx))(\lambda fx.f^n x) \to_\beta \lambda fx.f((\lambda fx.f^n x)fx) \to_\beta \lambda fx.f((\lambda x.f^n x)x)$$
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Definitions

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Lemmata

- $\text{zero?}\ \underline{n} \to_\beta^* \begin{cases} \text{T} & \text{if } n = 0 \\ \text{F} & \text{if } n > 0 \end{cases}$ $\qquad$ $\text{add}\ \underline{n}\ \underline{m} \to_\beta^* \underline{n+m}$ $\qquad$ $\text{mul}\ \underline{n}\ \underline{m} \to_\beta^* \underline{n \cdot m}$

Definition

$\mathsf{pred} \equiv \lambda n.n(\lambda uv.v(u\,\mathsf{succ}))(\lambda z.\underline{0})(\lambda z.z)$

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Proof

- $\text{pred}\ \underline{0} \to_\beta \underline{0}(\lambda uv.v(u\,\text{succ}))(\lambda z.\underline{0})(\lambda z.z) \to_\beta^* (\lambda z.\underline{0})(\lambda z.z) \to_\beta \underline{0}$
- $\text{pred}\ \underline{n+1} \to_\beta^* \underline{n}$ (homework exercise)

Example

representing factorial function in lambda calculus

$$\mathsf{fac}\, n = \mathsf{ite}\,(\mathsf{zero?}\, n)\, \underline{1}\, (\mathsf{mul}\, n\, (\mathsf{fac}\, (\mathsf{pred}\, n)))$$

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$$\mathsf{fac}\, \underline{2} \to_\beta^* F\, \mathsf{fac}\, \underline{2}$$

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$$\begin{aligned}\mathsf{fac}\,\underline{2} &\to_\beta^* F\,\mathsf{fac}\,\underline{2}\\ &\to_\beta^* \mathsf{ite}\,(\mathsf{zero?}\,\underline{2})\,\underline{1}\,(\mathsf{mul}\,\underline{2}\,(\mathsf{fac}\,(\mathsf{pred}\,\underline{2})))\end{aligned}$$

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\mathsf{fac}\; \underline{2} &\rightarrow_\beta^* F\; \mathsf{fac}\; \underline{2} \\
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&\to_\beta^* \cdots \\
&\to_\beta^* \mathsf{mul}\ \underline{2}\ (\mathsf{mul}\ \underline{1}\ (\mathsf{fac}\ \underline{0}))
\end{aligned}$$

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&\to_\beta^* \cdots \\
&\to_\beta^* \mathsf{mul}\ \underline{2}\ (\mathsf{mul}\ \underline{1}\ (\mathsf{fac}\ \underline{0})) \to_\beta^* \mathsf{mul}\ \underline{2}\ (\mathsf{mul}\ \underline{1}\ \underline{1})
\end{aligned}$$

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&\to_\beta^* \cdots \\
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\end{aligned}$$

Definition

partial function $f\colon \mathbb{N}^n \to \mathbb{N}$ is λ–definable if $\exists$ combinator F such that

$$f(x_1, \ldots, x_n) = y \qquad \Longrightarrow \qquad F\, \underline{x_1} \cdots \underline{x_n} \to_\beta^* \underline{y}$$

for all $x_1, \ldots, x_n, y \in \mathbb{N}$

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Theorem

partial recursive functions are λ–definable

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Proof

- zero function $\quad z(x) = 0$

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Theorem

partial recursive functions are λ–definable

Proof

- zero function $z(x) = 0$ zero $\equiv \lambda x.\underline{0}$
- successor function $s(x) = x + 1$ succ $\equiv \lambda n f x. f(n f x)$

Definition

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Proof (cont'd)

- composition $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$

Proof (cont'd)

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$$F \equiv \lambda \vec{x}.\, G\,(H_1\,\vec{x}) \cdots (H_m\,\vec{x})$$

- primitive recusion $$f(0, \vec{y}) = g(\vec{y})$$ $$f(x+1, \vec{y}) = h(f(x, \vec{y}), x, \vec{y})$$

Proof (cont'd)

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$$F = \lambda x\,\vec{y}.\, \mathsf{ite}\,(\mathsf{zero?}\,x)\,(G\,\vec{y})\,(H\,(F\,(\mathsf{pred}\,x)\,\vec{y})\,(\mathsf{pred}\,x)\,\vec{y})$$

Proof (cont'd)

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$$= \Theta\,(\lambda f\,x\,\vec{y}.\, \mathsf{ite}\,(\mathsf{zero?}\,x)\,(G\,\vec{y})\,(H\,(f\,(\mathsf{pred}\,x)\,\vec{y})\,(\mathsf{pred}\,x)\,\vec{y}))$$

Proof (cont'd)

- minimization $f(\vec{x}) = (\mu\, i)\,(g(i, x_1, \ldots, x_n) = 0)$

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Theorem

λ–definable function are partial recursive

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Theorem

λ–definable function are partial recursive

Remark

however, cf. slide 28 of lecture 8 and slides 19–21 of lecture 9

Outline

Theorem

β–reduction has Church–Rosser property

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Corollary

$x \neq_{\beta} \lambda y.xy$

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Theorem

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Corollary

λ–calculus is consistent

Remark

x and $\lambda y.xy$ are extensionally equivalent: $xM =_\beta (\lambda y.xy)M$ for all $M \in \Lambda$

Definition

one-step η–reduction is smallest relation $\to_\eta$ on lambda terms satisfying

$$(\eta) \qquad \frac{x \notin \mathsf{FV}(M)}{\lambda x.Mx \to_\eta M}$$

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Theorem

$\beta\eta$–reduction has Church-Rosser property

Outline

1. **Summary of Previous Lecture**
2. **β–Reduction**
3. **Church–Rosser Theorem**
4. **λ–Definability**
5. **η–Reduction**
6. **Normalization Theorem**
7. **Test Practice**
8. **Summary**

Example

- $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$

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Question

how to compute β–normal forms (or $\beta\eta$–normal forms) ?

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Normalization Theorem

leftmost reduction strategy is normalizing

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- 15:15 – 18:00 in HS 10
- online registration required before 10 am on January 23
- closed book

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test practice on January 22

Outline

Important Concepts

- α–conversion
- β–conversion
- β–reduction
- Church–Rosser theorem
- η–reduction
- lambda calculus
- λ–definability
- normalization theorem

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homework for January 22