universität innsbruck

WS 2023 lecture 14

Computability Theory

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Outline

Definition

set of lambda terms (Λ) is built from

- infinite set of variables $\mathcal{V} = \{x, y, z, \dots\}$ $\quad x \in \mathcal{V} \implies x \in \Lambda$
- application $\quad M, N \in \Lambda \implies (MN) \in \Lambda$
- abstraction $\quad x \in \mathcal{V}, M \in \Lambda \implies (\lambda x.M) \in \Lambda$

Notation

$M \equiv N$ if M and N are identical

Definition

- set $\mathrm{FV}(M)$ of free variables of lambda term M is inductively defined:

$$\mathrm{FV}(x) = \{x\} \qquad \mathrm{FV}(MN) = \mathrm{FV}(M) \cup \mathrm{FV}(N) \qquad \mathrm{FV}(\lambda x.M) = \mathrm{FV}(M) \setminus \{x\}$$

- lambda term M is closed (or combinator) if $\mathrm{FV}(M) = \varnothing$

Definition (Renaming)

$$x\{y/x\} \equiv y \qquad z\{y/x\} \equiv z \quad \text{if } x \neq z$$

$$(MN)\{y/x\} \equiv (M\{y/x\})(N\{y/x\})$$

$$(\lambda x.M)\{y/x\} \equiv \lambda y.(M\{y/x\}) \qquad (\lambda z.M)\{y/x\} \equiv \lambda z.(M\{y/x\}) \quad \text{if } x \neq z$$

Definition

α–equivalence is smallest congruence relation $\equiv_\alpha$ on lambda terms such that

$$\lambda x.M \equiv_\alpha \lambda y.(M\{y/x\})$$

for all terms M and variables y that do not occur in M

Convention

lambda terms are identified up to α–equivalence

Definition

$M[N/x]$ denotes result of substituting N for free occurrences of x in M:

$$x[N/x] \equiv N \qquad y[N/x] \equiv y \quad \text{if } x \neq y$$
$$(MP)[N/x] \equiv (M[N/x])(P[N/x])$$
$$(\lambda x.M)[N/x] \equiv \lambda x.M$$
$$(\lambda y.M)[N/x] \equiv \lambda y.(M[N/x]) \quad \text{if } x \neq y \text{ and } y \notin \mathsf{FV}(N)$$
$$(\lambda y.M)[N/x] \equiv \lambda z.(M\{z/y\}[N/x]) \quad \text{if } x \neq y,\ y \in \mathsf{FV}(N) \text{ and } z \text{ is fresh}$$

Definition

one-step β–reduction is smallest relation $\to_\beta$ on lambda terms satisfying

$$(\beta)\ \frac{}{(\lambda x.M)N \to_\beta M[N/x]} \qquad \frac{M \to_\beta M'}{MN \to_\beta M'N}\ (\text{congruence})$$

$$(\xi)\ \frac{M \to_\beta M'}{\lambda x.M \to_\beta \lambda x.M'} \qquad \frac{N \to_\beta N'}{MN \to_\beta MN'}\ (\text{congruence})$$

Definitions

- β–normal form is lambda term without β–redexes
- β–conversion ($=_\beta$) is transitive symmetric reflexive closure of $\to_\beta$

Definition

lambda term N is fixed point of lambda term F if $FN =_\beta N$

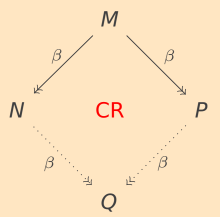
Theorem

every lambda term has fixed point

Definition (Turing's Fixed Point Combinator)

$\Theta \equiv AA$ with $A = \lambda xy.y(xxy)$

Church–Rosser Theorem



Definition (Parallel Reduction Revisited)

$$\frac{}{M \mathrel{\circ\!\!\to}_\beta M} \qquad \frac{M \mathrel{\circ\!\!\to}_\beta M' \quad N \mathrel{\circ\!\!\to}_\beta N'}{(\lambda x.M)N \mathrel{\circ\!\!\to}_\beta M'[N'/x]} \qquad \frac{M \mathrel{\circ\!\!\to}_\beta M'}{\lambda x.M \mathrel{\circ\!\!\to}_\beta \lambda x.M'} \qquad \frac{M \mathrel{\circ\!\!\to}_\beta M' \quad N \mathrel{\circ\!\!\to}_\beta N'}{MN \mathrel{\circ\!\!\to}_\beta M'N'}$$

Lemma

$\to_\beta \subseteq \mathrel{\circ\!\!\to}_\beta \subseteq \to_\beta^*$

Definition

M^* is maximal parallel one-step reduct of M:

① $x^* = x$

② $(PN)^* = P^*N^*$ if PN is no β–redex

③ $((\lambda x.Q)N)^* = Q^*[N^*/x]$

④ $(\lambda x.N)^* = \lambda x.N^*$

Lemma

if $M \mathrel{\circ\!\!\to}_\beta N$ then $N \mathrel{\circ\!\!\to}_\beta M^*$

Lemma (Diamond Property of Parallel Reduction)

$\forall\, M, N, P \in \Lambda$ such that $M \mathrel{\circ\!\!\to}_\beta N$ and $M \mathrel{\circ\!\!\to}_\beta P$
$\exists\, Q \in \Lambda$ such that $N \mathrel{\circ\!\!\to}_\beta Q$ and $P \mathrel{\circ\!\!\to}_\beta Q$

Definition (Church Numerals)

for every natural number n

$$\underline{n} \equiv \lambda f x. f^n x \quad \text{where} \quad F^n M \equiv \begin{cases} M & \text{if } n = 0 \\ F(F^{n-1}M) & \text{if } n > 0 \end{cases}$$

Definition

partial function $f \colon \mathbb{N}^n \to \mathbb{N}$ is λ–definable if $\exists$ combinator F such that

$$f(x_1, \dots, x_n) = y \quad \Longrightarrow \quad F\,\underline{x_1} \cdots \underline{x_n} \to_\beta^* \underline{y}$$

$$f(x_1, \dots, x_n) \text{ is undefined} \quad \Longrightarrow \quad F\,\underline{x_1} \cdots \underline{x_n} \text{ is not normalizing}$$

for all $x_1, \dots, x_n, y \in \mathbb{N}$

Theorem

function $f \colon \mathbb{N}^n \to \mathbb{N}$ is partial recursive $\iff$ f is λ–definable

Definitions

- one-step η–reduction is smallest relation $\to_\eta$ on lambda terms satisfying

$$(\eta)\ \frac{x \notin \mathrm{FV}(M)}{\lambda x. M x \to_\eta M} \qquad \frac{M \to_\eta M'}{M N \to_\eta M' N}\ \text{(congruence)}$$

$$(\xi)\ \frac{M \to_\eta M'}{\lambda x. M \to_\eta \lambda x. M'} \qquad \frac{N \to_\eta N'}{M N \to_\eta M N'}\ \text{(congruence)}$$

- one-step $\beta\eta$–reduction $\to_{\beta\eta}$ is union of $\to_\beta$ and $\to_\eta$

Theorem

$\beta\eta$–reduction has Church-Rosser property

Normalization Theorem

leftmost reduction strategy is normalizing

Outline

Recommended Courses

- WM 1: Automata and Logic (WS 2024)
- WM 2: Constraint Solving (SS 2024)
- WM 9: Research Seminar CL / TCS (SS 2024)
- WM 7: Interactive Theorem Proving (SS 2024)
- WM 7: Interactive Theorem Proving in Isabelle / HOL (SS 2024)
- WM 8: Automated Theorem Proving (SS 2024)
- WM 8: Quantum Computation (SS 2024)
- WM 20: Term Rewriting (Obergurgl)
- WM 20: Lambda Calculus and Type Theory (Obergurgl)
- Track C of ISR 2024 (Obergurgl)

Master Specializations

https://www.uibk.ac.at/informatik/master-computer-science/

Master Projects

http://cl-informatik.uibk.ac.at/teaching/master/available.php

$ 20,000 Prize: S Combinator Challenge

. . . for proving — or disproving — that the *S* combinator alone can support universal computation

evaluation 2024

Outline

1. Summary of Previous Lecture
2. Recommendations
3. **Test Practice**

Test on January 29

- 15:15 – 18:00 in HS 10
- online registration required before 10 am on January 23
- closed book
- score $= \min(\max(\frac{2}{3}(E+P)+\frac{1}{3}T+B, T+B), 100)$

Earlier Exams / Tests

- SS 2022 (test)
- WS 2017 – 2
- WS 2017 – 1
- WS 2014 – 2
- WS 2014 – 1
- SS 2012
- SS 2008 – 2
- SS 2008 – 1
- SS 2007
- SS 2006 – 2
- SS 2006 – 1
- WS 2004