
A simple visual proof of the Schröder-Bernstein theorem

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1 Introduction

The Schröder-Bernstein theorem is the critical step in the demonstration that the cardinal numbers are ordered. The theorem asserts:

Let A and B be sets. Let f be an injective mapping of A into B and g be an injective mapping of B into A . Then there exists a bijective mapping h of A into B .

The purpose of this article is to show *explicitly* how to construct such a bijective mapping h in a very simple *visual* way that is much easier to remember than any of the previous proofs. That is because the construction of h in the general case is *exactly analogous* to the construction in the following special case.

Special case. $A = B = \{0, 1, 2, 3, \dots, \omega\}$ and $f(x) = x + 1$, $g(x) = x + 1$. That is, $f(0) = 1$, $f(1) = 2$, $f(2) = 3, \dots$, $f(\omega) = \omega$, and $g(0) = 1$, $g(1) = 2$, $g(2) = 3, \dots$, $g(\omega) = \omega$.

Das Schröder-Bernstein Theorem besagt, dass eine Bijektion zwischen zwei Mengen A und B existiert, sobald A injektiv nach B und B injektiv nach A abgebildet werden kann. Obwohl diese Behauptung nahezu trivial erscheint, ist ihr Beweis keineswegs offensichtlich. Man findet verschiedene Beweise dieses Theorems in beinahe jedem Algebra-Lehrbuch. Viele dieser Beweise sind jedoch nach der persönlichen Meinung des Autors nicht leicht verständlich, insbesondere für Studienanfänger. In diesem Aufsatz gibt der Autor deshalb einen einfachen anschaulichen Beweis, von dem er hofft, dass man ihn nie wieder vergisst, wenn man ihn einmal verstanden hat.

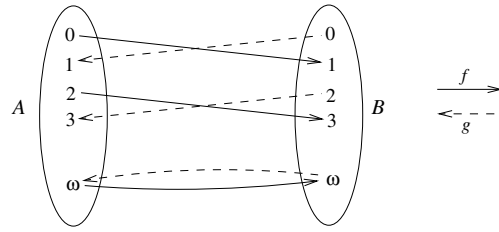


Fig. 1 Construction of h in the special case

In Fig. 1, we show how to construct a bijective mapping h such that for any $a \in A$, either $h(a) = f(a)$ or $h(a) = g^{-1}(a)$ as follows: $h(2i) = f(2i) = 2i + 1$, $h(2i + 1) = g^{-1}(2i + 1) = 2i$, $h(\omega) = f(\omega) = g^{-1}(\omega) = \omega$.

2 The proof

Similar to the special case, we now show *explicitly* how to construct a bijective mapping h such that for any $a \in A$, either $h(a) = f(a)$ or $h(a) = g^{-1}(a)$, in the general case.

Let $A_0 = A$, $B_0 = B$. For each natural number i , let $A_{i+1} = g(B_i)$ and $B_{i+1} = f(A_i)$. Then $A_0 \supset A_1 \supset A_2 \supset \dots$ and $B_0 \supset B_1 \supset B_2 \supset \dots$.

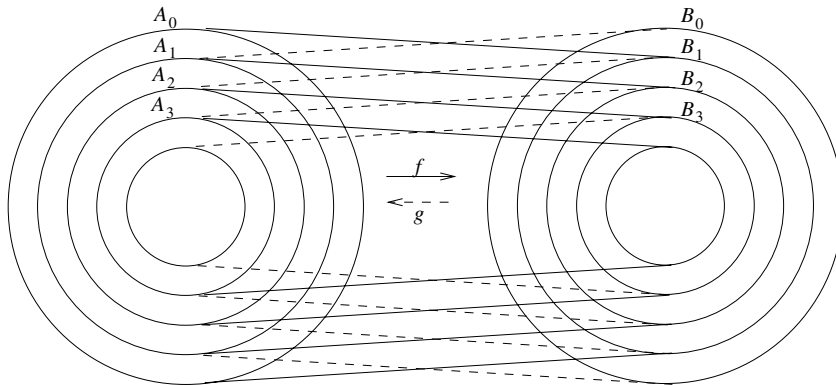


Fig. 2 $A_0 \supset A_1 \supset A_2 \supset \dots$ and $B_0 \supset B_1 \supset B_2 \supset \dots$

Let $C_0 = A_0 \setminus A_1$, $C_1 = A_1 \setminus A_2$, ..., and $C_\omega = \bigcap A_i$. Then A is partitioned into the following disjoint subsets $C_0, C_1, \dots, C_\omega$. Let $D_0 = B_0 \setminus B_1$, $D_1 = B_1 \setminus B_2$, ..., and $D_\omega = \bigcap B_i$. Then B is partitioned into the following disjoint subsets $D_0, D_1, \dots, D_\omega$. Clearly, f maps C_i bijectively to D_{i+1} and g maps D_i bijectively to C_{i+1} .

Furthermore, since $D_\omega \subset f(A)$ and

$$f^{-1}(D_\omega) = f^{-1}\left(\bigcap B_{i+1}\right) = \bigcap f^{-1}(B_{i+1}) = \bigcap A_i = C_\omega,$$

f maps C_ω bijectively to D_ω . Similarly, g maps D_ω bijectively to C_ω .

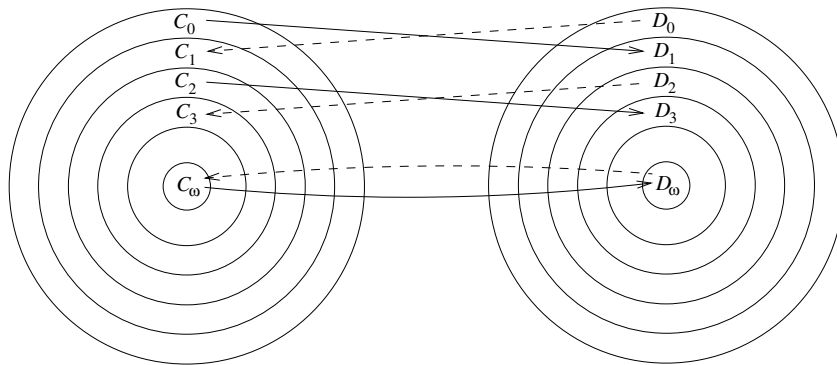


Fig. 3 Construction of h in the general case

The construction of $h : A \rightarrow B$ is depicted in Fig. 3: h maps C_{2i} bijectively to D_{2i+1} as f does, h maps C_{2i+1} bijectively to D_{2i} as g^{-1} does, and h maps C_ω bijectively to D_ω as f (or g^{-1}) does.

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