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Functional Programming

Week 11 – Lazy Evaluation, Infinite Lists

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Last Lecture

- `IO a` is type of I/O-actions with resulting type `a`
- `do`-blocks are used for sequential composition of I/O-actions
- clear separation between purely functional and I/O-code:
  - embed functional code into I/O: `return :: a -> IO a`
  - the other direction is not available: no function of type `IO a -> a`
- `ghc` compiles programs that provide `main :: IO ()` function in module `Main`
- example application: connect four
  - user-interface: I/O-code
  - game logic: purely functional

Monads

- bind (>>=), `return`, and `do`-notation are not restricted to I/O
- there exists a more general concept of monads
- example: also the `Maybe`-type is a monad

```
data Expr = Const Double | Div Expr Expr
eval :: Expr -> Maybe Double
eval (Const c) = return c
eval (Div expr1 expr2) = do
  x1 <- eval expr1
  x2 <- eval expr2
  if x2 == 0
    then Nothing
    else return (x1 / x2)
```

- monads won't be covered here, but they are the reason why the Haskell literature speaks about the I/O-monad

Evaluation Strategies

Pure Functions

- a function is pure if it always returns same result on same input
- pure functions are similar to mathematical functions
- examples of pure functions
  - addition
  - sort a list
  - ...
- examples of non-pure functions
  - roll a dice
  - current time
  - position of cursor
  - ...
- pure languages permit to define only pure functions
- Haskell is a pure language

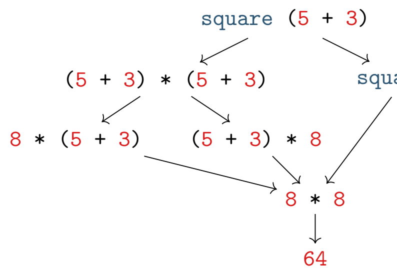
Pure Functions and I/O

- even I/O is pure in Haskell
- consider `main = getLine >>= putStrLn . ("Hello " ++)`
- it seems that the result depends on user input, so is not pure
- however `main :: IO ()`, so the functional value of `main` is not what is entered and printed during execution, but the value is of type `IO ()`, i.e., a sequence of actions that are executed when running the program; and indeed this sequence is always the same:

  first read some input i and then print the string `"Hello` i`"`
- alternative argumentation: interpret type `IO a` a state transformer on the outside world, e.g., as a function of type `RealWorld -> (RealWorld, a)`
- remark: in the remainder of this lecture we will only consider purely functional programs without I/O

Evaluation Order

- there are several ways to evaluate expressions, consider `square x = x * x`



- in pure languages, the evaluation order has no impact on resulting normal form
- normal form: an expression that cannot be evaluated further, a result

Theorem

Whenever there are two (different) ways to evaluate a Haskell expression to normal form, then the resulting normal forms are identical.

Standard Evaluation Strategies

- each functional language fixes the evaluation order via some evaluation strategy
- three prominent evaluation strategies (expressions represented as trees and dags)
  - call-by-value / strict / innermost: first evaluate arguments

```
               square    square       *
                  |         |        / \
square (5+3) =    +    =    8    =  8   8   =   64
                 / \
                5   3
```

  - call-by-name / non-strict / outermost: directly replace function application by rhs

```
               square      *          *         *
                  |       / \        / \       / \
square (5+3) =    +    = +   +   =  +   8  =  8   8  =  64
                 / \    / \ / \    / \
                5   3  5  3 5  3  5   3
```

  - call-by-need / lazy evaluation: like call-by-name + sharing (dags = directed acyclic graphs)

```
               square     *         *
                  |      ( )       ( )
square (5+3) =    +   =   +    =    8    =  64
                 / \     / \
                5   3   5   3
```

Evaluation Strategy of Haskell

- Haskell uses lazy evaluation with left-to-right argument order
- sharing is applied whenever a variable occurs multiple times
- example: consider definition `f x = g x + g (3 + 5) + x`
  - when evaluating `f (1 + 2) = g (1 + 2) + g (3 + 5) + (1 + 2)` the two occurrences of `1 + 2` are shared: they use the same variable `x`
  - when evaluating `f (3 + 5) = g (3 + 5) + g (3 + 5) + (3 + 5)` the two occurrences of `g (3 + 5)` are not shared: it was a coincidence that `x` was substituted by `3 + 5` and this equality is not detected at runtime
- there might be further sharing (depending on the compiler), e.g. sharing common subexpressions such as the expression `g x` in a function definition `f x = g x + h (g x)`
- argument evaluation within function invocation `f expr1 ... exprN` is mainly triggered by pattern matching, i.e., the process of finding the suitable defining equation `f pat1 ... patN = expr`, cf. slides 13 and 15 of week 3
- many builtin arithmetic functions will trigger evaluation of all arguments, e.g., `(0 :: Integer) * undefined` will result in error, and not in `0`

Evaluation Strategy and Termination

- consider the following Haskell script

```
three :: Integer -> Integer
three x = 3

inf :: Integer
inf = 1 + inf
```

- strict evaluation does not terminate, i.e., it will evaluate forever
  `three inf = three (1 + inf) = three (1 + (1 + inf)) = ...`
- non-strict and lazy evaluation are immediately done
  `three inf = 3`

Theorem

- if the evaluation of an expression terminates for some evaluation strategy, then it terminates using non-strict or lazy evaluation
- if the evaluation of an expression terminates using strict evaluation, then it terminates for every evaluation strategy

Comparison of Evaluation Strategies

- call-by-value
  - easy to understand
  - easy to implement
  - overhead in evaluating non-required expressions
  - used in many functional programming languages
- lazy evaluation
  - harder to understand
  - single evaluation step is more complicated to implement:
    pass arguments that are unevaluated expressions (thunks) instead of just values
  - overhead in computing with thunks
  - allows programmers to naturally define and work with infinite data
  - used in Haskell

Tail Recursion and Strict Evaluation

Different Kinds of Recursion

- a function calling itself is recursive
- functions that mutually call each other are mutually recursive

```
even n | n == 0     = True
       | otherwise = odd (n - 1)
odd n  | n == 0     = False
       | otherwise = even (n - 1)
```

- nested recursion: recursive calls inside recursive calls

```
ack n m | n == 0 = m + 1
        | m == 0 = ack (n - 1) 1
        | otherwise = ack (n - 1) (ack n (m - 1))
```

- linear recursion: at most one recursive call (per if-then-else branch)
  - `fib n | n >= 2 = fib (n - 1) + fib (n - 2)` ✘
  - `length (x : xs) = 1 + length xs` ✔
  - ``f x = if even x then f (x `div` 2) else f (3 * x + 1)`` ✔
- tail recursion and guarded recursion will be discussed in more detail

Tail Recursion

- tail recursion is special form of linear recursion
- additional requirement
  - recursive function calls happen at the outermost level
  - however, they can be within an if-then-else
- examples
  - `length (x : xs) = 1 + length xs` ✘
  - ``f x = if even x then f (x `div` 2) else f (3 * x + 1)`` ✔
- advantage of tail recursion
  - no dangling function calls
  - can be evaluated as loop
  - space efficient

Example: Advantage of Tail Recursion

- linear but not tail recursive variant

```
sumRec 0 = 0
sumRec n = n + sumRec (n - 1)

  sumRec 5 = 5 + sumRec (5 - 1)
= 5 + sumRec 4 = 5 + (4 + sumRec (4 - 1))
= 5 + (4 + sumRec 3) = 5 + (4 + (3 + sumRec (3 - 1)))  = ...
= 5 + (4 + (3 + (2 + (1 + 0)))) = ... = 15   -- linear space
```

- tail recursive variant using accumulator to store intermediate results

```
sumTr n = aux 0 n where
  aux acc 0 = acc
  aux acc n = aux (acc + n) (n - 1)

  sumTr 5
= aux 0 5 = aux (0 + 5) (5 - 1)
= aux 5 4 = aux (5 + 4) (4 - 1)
= aux 9 3 = ... = 15
 -- constant space, implement as loop with two variables: acc and n
```


Problem of Tail Recursion using Lazy Evaluation

```
sumTr n = aux 0 n where
  aux acc 0 = acc
  aux acc n = aux (acc + n) (n - 1)
```

- example evaluation of `sumTr` on previous slide used call-by-value
- in lazy evaluation `acc` and `n` are only evaluated on demand
- causes linear memory consumption in `sumTr`

```
  sumTr 5                   -- with lazy evaluation
= aux 0 5
= aux (0 + 5) (5 - 1)
= aux (0 + 5) 4
= aux ((0 + 5) + 4) (4 - 1)
= ...
= aux (((((0 + 5) + 4) + 3) + 2) + 1) 0
= ((((0 + 5) + 4) + 3) + 2) + 1 = ... = 15
```

Enforcing Evaluation

- Haskell function to enforce evaluation: `seq :: a -> b -> b`
- evaluation of `seq x y` first evaluates `x` to WHNF and then returns `y`
- WHNF: weak head normal form
- expression `e` is in WHNF iff it has one of the following three shapes
  - `e = C expr1 ... exprN` for some constructor `C` (constructor application)
  - `e = f expr1 ... exprN` if the defining equations of `f` have `M` > `N` arguments, i.e., they are of the form `f pat1 ... patM = expr` (too few arguments)
  - `e = \ pat1 ... patN -> expr` (λ-abstraction)
- examples
  - in WHNF: `True`, `7.1`, `(5+1) : [1] ++ [2]`, `(:)`, `undefined : undefined`, `(++)`, `(++ undefined)`, `\ x -> undefined`
  - not in WHNF: `[1] ++ [2]`, `(\ x -> x + 1) (1 + 2)`, `undefined ++ undefined`
  - evaluation: `let x = 1 + 2 in seq x (f x)`

```
= seq (1 + 2) (f (1 + 2))                   -- with 1 + 2 shared
= seq 3 (f 3)              -- seq enforced evaluation of argument
= f 3 = ...                        -- evaluation of f 3 continues
```


Example Application using `seq`

- solve memory problem in tail recursion by enforcing evaluation of accumulator

```
sumTrSeq n = aux 0 n where
  aux acc 0 = acc
  aux acc n = let accN = acc + n in seq accN (aux accN (n - 1))
```

```
  sumTrSeq 5
= aux 0 5
= let accN = 0 + 5 in seq accN (aux accN (5 - 1))
= seq (0 + 5) (aux (0 + 5) (5 - 1))              -- 0 + 5 is shared
= seq 5 (aux 5 (5 - 1))                            -- and evaluated
= aux 5 (5 - 1)
= aux 5 4                      -- pattern matching triggers evaluation
= let accN = 5 + 4 in seq accN (aux accN (4 - 1))
= seq (5 + 4) (aux (5 + 4) (4 - 1))              -- 5 + 4 is shared
= seq 9 (aux 9 (4 - 1))                            -- and evaluated
= aux 9 (4 - 1)                              -- same structure as above
= ... = 15                                         -- constant space
```


Enforcing Strict Evaluation ... Continued

- besides `seq`, there are other options to enforce strict evaluation
- strict library functions like a strict version of `foldl`:
  `Data.List.foldl' :: (b -> a -> b) -> b -> [a] -> b`

```
import Data.List
length = foldl' (\ x _ -> x + 1) 0
```

- pattern matching with bang patterns to enforce evaluation, e.g.,
  `aux acc n = let !accN = acc + n in aux accN (n - 1)`
- strict datatypes
- see https://downloads.haskell.org/~ghc/9.2.5/docs/html/users_guide/exts/strict.html for further details

Lazy Evaluation and Infinite Lists

Guarded Recursion

- every recursive call is inside ("guarded by") a constructor
- also known as "tail recursion modulo cons"
- more important than tail recursion in Haskell
- allows the result to be consumed lazily – tail recursion provides the result only at the end
- examples
  - ```
    map f [] = []
    map f (x:xs) = f x : map f xs                ✔
    ```
  - ```
    reverse xs = revAux xs [] where
      revAux [] ys = ys                          ✘
      revAux (x : xs) ys = revAux xs (x : ys)
    ```
  - ```
    enumFrom x = x : enumFrom (x + 1)            ✔
    ```
- remarks on `enumFrom`
  - above definition is simplified, actual definition works for members of type class `Enum`, e.g., `Int`, `Char`, `Integer`, `Double`, ... and prevents overflows
  - syntactic sugar: `[x..] = enumFrom x`

Infinite Lists

- infinite lists ∼ sequences of elements (also known as streams)
- programming with infinite lists: producing and consuming elements of sequences one after another (e.g., with guarded recursion)
- example: `[x..] = x : [x + 1 ..]` generates infinite list
- in combination with lazy evaluation, infinite lists do not always cause non-termination
- examples

```
  take 2 [7..]
= take 2 (7 : [8..])
= 7 : take 1 [8..]
= 7 : 8 : take 0 [9..]
= [7, 8]

  takeWhile (< 95) $ map (\ x -> x * x) [0..]
= ... = [0,1,4,9,16,25,36,49,64,81]

  filter (< 100)   $ map (\ x -> x * x) [0..]
= ... = [0,1,4,9,16,25,36,49,64,81  -- interrupted
```


Laziness and Infinite Data Structures Facilitate Modularity

- separation of concerns
  - write small functions with specific tasks
  - use potentially infinite data structures
- example: find index of first list element satisfying predicate
  - function `firstIndex :: (a -> Bool) -> [a] -> Int`
  - in Haskell
    `firstIndex p = fst . head . filter (p . snd) . zip [0..]`
  - (lazy) evaluation (without showing expansion of (.) and ($))
    ```
      firstIndex (== 1) [1..9]
    = fst . head . filter ((== 1) . snd) $ zip [0..] [1..9]
    = fst . head . filter ((== 1) . snd) $ (0,1) : zip [1..] [2..9]
    = fst . head $ (0,1) : filter ((== 1) . snd) $ zip [1..] [2..9]
    = fst (0,1)
    = 0
    ```
  - without laziness several complete list traversals are required when using library functions (e.g., computation of length and addition of indices)
  - remark: `firstIndex` works for arbitrary lists as input: finite and infinite

Sieve of Eratosthenes

- goal: generate list of all prime numbers
- algorithm
  1. start with list of all natural numbers (from 2 on)
  2. mark first element x as prime
  3. remove all multiples of x
  4. go to Step 2
- in Haskell

```
primes :: [Integer]
primes = sieve [2..] where
  sieve (x : xs) = x : sieve (filter (\ y -> y `mod` x /= 0) xs)

> take 1000 primes          -- the first 1000 primes
> takeWhile (< 1000) primes -- all primes below 1000
```
```

Summary

- in pure functional languages such as Haskell the result does not depend on the evaluation strategy
- different kinds of recursion
- tail recursion is usually efficient as it can be implemented as loop
- `seq` can be used to enforce strict evaluation (in particular of accumulators)
- lazy evaluation allows modeling of infinite lists
- guarded recursion is important for algorithms on infinite lists
- infinite lists permit to naturally formulate several algorithms
  (without having to take care of boundary conditions)