universität innsbruck

WS 2023/2024

Functional Programming

Week 12 – Cyclic Data Structures, Abstract Data Types

René Thiemann James Fox Lukas Hofbauer Christian Sternagel Tobias Niederbrunner

Department of Computer Science

Last Lecture – Evaluation Strategies

- evaluation strategies determine order of evaluation
- three kinds: innermost, outermost, and lazy evaluation (outermost + sharing)
- in pure functional languages the result does not depend on the evaluation strategy
  - consider non-pure language with function `uNum :: Int` that asks the user for a number and returns it
  - what is result of evaluating
    `f uNum where f x = x - x`
    if the user will enter the two numbers 5 and 3?
  - outermost (left-to-right): `f uNum = uNum - uNum = 5 - uNum = 5 - 3 = 2`
  - outermost (right-to-left): `f uNum = uNum - uNum = uNum - 5 = 3 - 5 = -2`
  - innermost: `f uNum = f 5 = 5 - 5 = 0`
- tail recursion in combination with innermost strategy can be implemented as loop
- `seq a b` enforces evaluation of `a` to WHNF and then results in `b`
  - pitfall: in the following Haskell program, `seq` does not have the required effect

```haskell
sumAux acc 0 = acc
sumAux acc n = let accN = acc + n in sumAux (seq accN accN) (n - 1)
-- correct:  = let accN = acc + n in seq accN (sumAux accN (n - 1))
```

Last Lecture – Lazy Evaluation and Infinite Data Structures

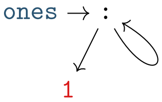
- it is possible to define infinite lists, trees, etc., e.g.,
  `enumFrom x = x : enumFrom (x + 1)`
- finite parts of infinite lists can be accessed, e.g., via `take`, `takeWhile`, etc., and lazy evaluation will not enforce computation of whole infinite list
- benefit: natural definition of several algorithms without having to worry about bounds, lengths, etc.
- main algorithmic structure: guarded recursion so that new constructors are produced in each recursive evaluation step

Cyclic Data Structures

Cyclic Lists

- aim: direct definition of infinite lists which are implicitly computed on demand via lazy evaluation
- methodology: provide start of cyclic list and remaining cyclic list
- a first example: the infinite list of ones
  - starting element is 1
  - remaining list is the list of ones itself
  - Haskell definition

    ```haskell
    ones :: [Integer]
    ones = 1 : ones
    ```
  - created cyclic data structure



Combination of Lists

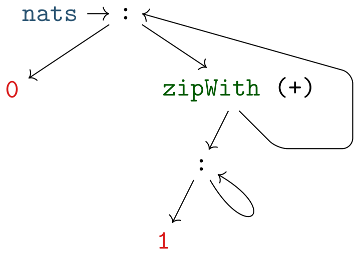
- cyclic definitions may involve auxiliary functions such as `map`, `filter`, and `zipWith`
- example: the list of natural numbers: `nats`
  - start is 0
  - remainder is addition of the list of ones with natural numbers itself

```
  0 1 2 3 4 5 ...
+ 1 1 1 1 1 1 ...
= 1 2 3 4 5 6 ...    (= tail nats)
```

  - in Haskell

```haskell
nats :: [Integer]
nats = 0 : zipWith (+) nats ones
```

  - created cyclic data structure:



Computing Fibonacci Numbers

- definition: $fib(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ fib(n-1) + fib(n-2), & \text{otherwise} \end{cases}$
- efficient computation of Fibonacci numbers via cyclic lists
- two starting elements: 0 and 1
- remainder is tail(tail fibs) = fibs + tail fibs

```
  0 1 1 2 3  5  8 ...   -- fibs
+ 1 1 2 3 5  8 13 ...   -- tail fibs
= 1 2 3 5 8 13 21 ...   -- tail (tail fibs)
```

- in Haskell

```haskell
fibs :: [Integer]
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
```

- remark: two starting elements, since otherwise `tail fibs` in rhs cannot be evaluated

Fibonacci Numbers in Haskell

- implementation was given in first lecture (slide 19 of week 1)

```
fibs :: [Integer]
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
```

- cyclic definition of list, evaluation:

```
  0 : 1 : zipWith (+) ● (tail ● )
= 0 : 1 : zipWith (+) ● ●
= 0 : 1 : zipWith (+) (0 : ● ) (1 : ● )
= 0 : 1 : 1 : zipWith (+) ● ●
= 0 : 1 : 1 : zipWith (+) (1 : ● ) (1 : ● )
= 0 : 1 : 1 : 2 : zipWith (+) ● ●
= 0 : 1 : 1 : 2 : zipWith (+) (1 : ● ) (2 : ● )
```

Infinite Data Structures Beyond Lists

- lists are not the only infinite data structure, e.g., there are also infinite trees (vertically and/or horizontally)
- also cyclic trees can be defined, e.g., consider a tree that represents all (finite and infinite) paths in the graph starting from node 1

$$\longrightarrow 1 \rightleftarrows 2 \rightleftarrows 3 \longrightarrow 4$$

- in Haskell we use a mutual recursive definition of four trees (`Paths`)

```haskell
data Paths = Root Integer [Paths]

paths1 = Root 1 [paths2]
paths2 = Root 2 [paths1, paths3]
paths3 = Root 3 [paths2, paths4]
paths4 = Root 4 []
```

- access finite parts of infinite tree in the same way as for infinite lists;
  example: analogy of "take first n elements of a potentially infinite list" would be a function for computing "all paths of length up to n of an potentially infinite tree"

Abstract Data Types

Concrete and Abstract Datatypes

- concrete datatypes
  - defined via `data` which defines values of that type
  - user defines own operations on this type via pattern matching
  - no need for primitive operations on that type
  - examples: `Rat`, `Person`, `Expr`, `Bool`, `[a]`, ...
- abstract datatypes
  - defined via their primitive operations
  - usually no access to internal structure of representation of values
  - pattern matching only via equality: `f 5 = ...` is equivalent to `f x = if x == 5 ...`
  - abstraction barrier: internal structure can be easily changed
  - meaning of operations usually specified
  - examples: `Char`, `Integer`, `Double`, ... which provide basic arithmetic operations and conversion to strings

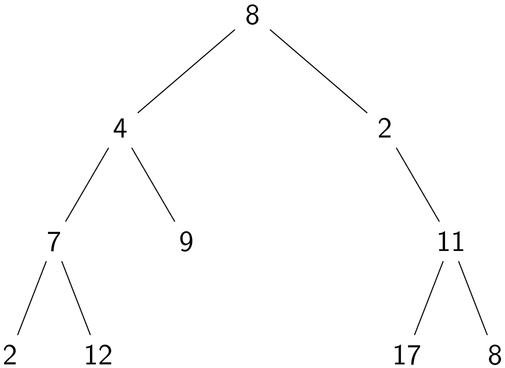
Example Abstract Datatype: Queues

- queues are useful in computer science: printer (jobs), web-server (requests), ...
- queue provides the following operations
  - `empty :: Queue a` – the empty queue for elements of type `a`
  - `isEmpty :: Queue a -> Bool` – check whether queue is empty
  - `dequeue :: Queue a -> (a, Queue a)` – remove head of queue
  - `enqueue :: a -> Queue a -> Queue a` – add new element to end of queue

  these operations in combination with their types are the signature of the abstract datatype `Queue a`
- signature only gives idea about operations; more information can be specified via axiomatic specification in the form of equations or formulas
  - `isEmpty empty`
  - `not $ isEmpty $ enqueue x q`
  - `dequeue (enqueue x empty) = (x, empty)`
  - `not $ isEmpty q` $\longrightarrow$ `dequeue q = (y, q')` $\longrightarrow$
    `dequeue (enqueue x q) = (y, enqueue x q')`

Example Application for Queues: Tree-Traversals

- consider binary tree



- tree-traversal: visit all nodes, e.g., to search for node, or convert nodes to list
  - in-order [2,7,12,4,9,8,2,17,11,8]
  - depth-first search, pre-order [8,4,7,2,12,9,2,11,17,8]
  - breadth-first search [8,4,2,7,9,11,2,12,17,8]

Tree Traversals in Haskell

```haskell
data Tree a = Empty | Node (Tree a) a (Tree a)

inOrder :: Tree a -> [a]
inOrder Empty = []
inOrder (Node l n r) = inOrder l ++ [n] ++ inOrder r

-- preOrder is similar to inOrder

bfs :: Tree a -> [a]
bfs t = bfsMain (enqueue t empty) where
  bfsMain :: Queue (Tree a) -> [a]
  bfsMain q
    | isEmpty q = []
    | otherwise = let (t', q') = dequeue q in
        case t' of
          Empty -> bfsMain q'
          Node l n r -> n : (bfsMain $ enqueue r $ enqueue l $ q')
```

Implementing an Abstract Datatype

- implementation has to provide the desired operations and must satisfy the specification (informal text or axiomatic)
  - `empty :: Queue a`
  - `isEmpty :: Queue a -> Bool`
  - `dequeue :: Queue a -> (a, Queue a)`
  - `enqueue :: a -> Queue a -> Queue a`
  - `isEmpty empty`
  - `not $ isEmpty $ enqueue x q`
  - `dequeue (enqueue x empty) = (x, empty)`
  - `not $ isEmpty q` $\longrightarrow$ `dequeue q = (y, q')` $\longrightarrow$
    `dequeue (enqueue x q) = (y, enqueue x q')`
- any implementation can be used, e.g., a basic one in the beginning, which might be replaced by more efficient one later on
- if corner cases are not specified, implementation can choose freely, e.g., how dequeue should behave on empty queues
- modules can be used to hide internals

A Basic Implementation of Queues

```haskell
module BasicQueue(Queue, empty, isEmpty, dequeue, enqueue) where

data Queue a = Empty | Enqueue a (Queue a)

empty = Empty
enqueue = Enqueue

isEmpty Empty = True
isEmpty (Enqueue x q) = False

dequeue (Enqueue x Empty) = (x, Empty)
dequeue (Enqueue x q) = (y, Enqueue x q') where
  (y, q') = dequeue q
dequeue Empty = error "dequeue on empty queue"
```

- implementation is rather direct translation of specification
- `empty` and `enqueue` are implemented as constructors of queues, and exported; still the constructors itself are not exported and so internal structure is not revealed, e.g., externally no pattern matching on queues is possible

Notes on the Basic Implementation of Queues

```haskell
...
data Queue a = Empty | Enqueue a (Queue a)

isEmpty Empty = True
isEmpty (Enqueue x q) = False

dequeue (Enqueue x Empty) = (x, Empty)
dequeue (Enqueue x q) = (y, Enqueue x q') where
  (y, q') = dequeue q
dequeue Empty = error "dequeue on empty queue"
```

- we did not prove that implementation meets the specification; will be covered in
  - program verification (bsc), or
  - interactive theorem proving (msc)
- implementation is inefficient, since first enqueuing n elements and then dequeueing n elements requires $\sim \frac{1}{2}n^2$ evaluation steps

Towards a More Efficient Implementation of Queues

- previous queue-type is essentially a list where the list head represents the end of the queue (queue = reversed list)
- assume customers 1, 2, 3 and 4 enqueue in that order, then the representation is `[4, 3, 2, 1]`
- enqueuing is efficient since it just adds element in front of list
- dequeuing is expensive since it traverses and rebuilds whole list
- new version: store queue as pair of two lists: (front, rear)
  - front part of queue (head of queue is head of list)
  - rear part of queue in reverse order (tail of queue is head of list)
  - invariant: whenever front part of queue is empty then whole queue is empty
- example queue with customers 1, 2, 3, 4 has multiple representations
  - `([1,2,3,4], [])` ✔
  - `([1,2,3], [4])` ✔
  - `([1], [4,3,2])` ✔
  - `([], [4,3,2,1])` ✘
- advantage: often constant time access to both ends of queue

More Efficient Implementation of Queues

```haskell
module BetterQueue(Queue, empty, isEmpty, dequeue, enqueue) where

type Queue a = ([a], [a])

empty :: Queue a
empty = ([], [])

isEmpty :: Queue a -> Bool
isEmpty (front, _) = null front

enqueue :: a -> Queue a -> Queue a
enqueue x (front, rear) = maybeMtf (front, x : rear)

dequeue :: Queue a -> (a, Queue a)
dequeue ([], _) = error "dequeue on empty queue"
dequeue (x : front, rear) = (x, maybeMtf (front, rear))

maybeMtf ([], rear) = (reverse rear, [])
maybeMtf q = q
```

Efficiency of More Efficient Implementation

```
dequeue ([], _) = error "dequeue on empty queue"
dequeue (x : front, rear) = (x, maybeMtf (front, rear))

maybeMtf ([], rear) = (reverse rear, [])
maybeMtf q = q
```

- move-to-front operation required when `front` is empty (obey invariant)
- single move-to-front operation may be expensive, but these operations are rare
- efficiency: n queue operations require at most $2n$ evaluation steps
- proving technique: amortized cost analysis, will be covered in course algorithms and data-structures

Abstraction Barrier of More Efficient Implementation

```haskell
module BetterQueue(Queue, empty, isEmpty, dequeue, enqueue) where

type Queue a = ([a], [a])
...
empty :: Queue a
...
```

- since `type` is just an abbreviation:
  `empty :: ([a], [a])`
- since pairs and lists are visible, external users can completely inspect internal structure and create queues which are not permitted, e.g., `isEmpty ([], [4,3,2,1])` evaluates to `True`
- since `type` is just an abbreviation, in particular `Queue`'s are instances of `Eq`, `Show`, and `Ord`, which might not be intended
- simple solution: hide representation in new datatype
  `data Queue a = Queue ([a], [a])`

Implementation with Separate Datatype

```haskell
module DataQueue(Queue, empty, isEmpty, dequeue, enqueue) where

data Queue a = Queue ([a], [a])                    -- new datatype

empty :: Queue a
empty = Queue ([], [])                  -- wrap Queue constructor around

isEmpty :: Queue a -> Bool
isEmpty (Queue (f, _)) = null f             -- unwrap Queue constructor

queue = Queue . maybeMtf

enqueue :: a -> Queue a -> Queue a
enqueue x (Queue (f, r)) = queue (f, x : r)

dequeue :: Queue a -> (a, Queue a)
dequeue (Queue ([], _)) = error "dequeue on empty queue"
dequeue (Queue (x : f, r)) = (x, queue (f, r))

maybeMtf ([], r) = (reverse r, [])
maybeMtf q = q
```

Newtype

```haskell
data Queue a = Queue ([a], [a])

queue = Queue . maybeMtf

enqueue :: a -> Queue a -> Queue a
enqueue x (Queue (f, r)) = queue (f, x : r)
...
```

- always wrapping and unwrapping the `Queue` constructor has some efficiency penalty
- more efficient version to hide an implementation type: newtype
- syntax: `newtype TName tvars = CName typ`
  - only one constructor (`CName`) allowed
  - this constructor must have exactly one argument type
  - nearly equivalent to `data TName tvars = CName typ`,
    one difference: newtype is faster (`CName` won't be created at runtime)
- minimal change in implementation of queues
  - `newtype Queue a = Queue ([a], [a])` instead of
    `data Queue a = Queue ([a], [a])`

Summary

- cyclic lists
  - implicit definition of infinite lists
  - can be used to elegantly and efficiently implement some functions (Fibonacci)
- abstract datatypes: specify operations with their properties;
  introduces abstraction barriers that permit change of implementations
- example: different implementations of queues
- newtype is efficient variant of **`data`** in case there is only one constructor with one argument
- example abstract datatypes
  - known: `Queue`, `Double`, `Char`, `Integer`, ...
  - further examples: sets (Data.Set), stacks (Data.Stack), dictionaries (Data.Map), ...