universität innsbruck

Machine Learning for Theorem Proving

Lecture 6 (VU)

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Topics

- Cropper, Morrel: Learning programs by learning from failures
- Gauthier, Olsák, Urban: Alien coding
- Johannson: Lemma Discovery for Induction a Survey
- Selsam et al: Learning a SAT Solver …
- Bartek, Suda: How Much Should Tyis Symbol Weigh?
- Suda: Vampire with a Brain Is a Good ITP Hammer
- Azerbayev et al: Autoformalizing and Formally Proving Undergraduate Math
- Mikula et al: Magnushammer, Transformer-based Approach to Premise Selection
- Chvalovsky et al: Guiding an Instantiation Prover with GNNs

Mini Test

λ-calculus

- $2 := \lambda f.\lambda x.f(f\,x)$
- $t := \lambda m.\lambda n.\lambda f.\lambda x.m(n\,f)\,x$
- $t\,2\,2 = ?$

Find functions of types

- $(i \rightarrow i) \rightarrow i \rightarrow i$
- $(\alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta$

Core ML

Definition (Expressions)

$$e ::= \overbrace{x \mid e\ e \mid \lambda x.e}^{\lambda\text{-Calculus}} \mid \underbrace{c}_{\text{primitives/constants}} \mid \underbrace{\textbf{let}\ x = e\ \textbf{in}\ e}_{\text{let binding}} \mid \underbrace{\textbf{if}\ e\ \textbf{then}\ e\ \textbf{else}\ e}_{\text{conditional}}$$

Primitives

Boolean: true, false, $<$, $>$, ...

Arithmetic: $\times$, $+$, $\div$, $-$, 0, 1, ...

Tuples: pair, fst, snd

Lists: nil, cons, hd, tl

What is Type Checking?

- Given some environment (assigning types to primitives)
- together with a core ML expression and a type
- check whether the expression really has that type
- (with respect to that environment)

Preliminaries

Definition (Types)

$$\tau ::= \underbrace{\alpha}_{\text{type variable}} \mid \overbrace{\tau \to \tau}^{\text{function type constructor}} \mid \underbrace{g(\tau, \ldots, \tau)}_{\text{data type constructor}}$$

Convention

- type variables α, α_0, α_1, ..., β, β_0, ...
- function type constructor '$\to$' is right associative
- base data type constructors: int, bool (instead of int(), bool())

Example

int $\to$ bool, (int $\to$ list(int)) $\to$ bool, list(α_0) $\to$ int, ...

Preliminaries (cont'd)

(Typing) environment E: maps (variables and) primitives to types

$(e : \tau) \in E$ *"e is of type τ in E"*

(note: parentheses around $e : \tau$ will be usually dropped)

(Typing) judgment:

$E \vdash e : \tau$ *"it can be proved that expression e has type τ in environment E"*

Example

- environment $P = \{+ : \mathrm{int} \to \mathrm{int} \to \mathrm{int}, \mathrm{nil} : \mathrm{list}(\alpha), \mathrm{true} : \mathrm{bool}, \ldots\}$
- judgement $P \vdash \mathrm{true} : \mathrm{bool}$
- judgement $P \nvdash \mathrm{true} : \mathrm{int}$

Convention

$E, e : \tau$ abbreviates $E \cup \{e : \tau\}$

The Type Checking System $\mathcal{C}$

$$\frac{}{E, e : \tau \vdash e : \tau} \text{ (ref)} \qquad \frac{E \vdash e_1 : \tau_2 \to \tau_1 \quad E \vdash e_2 : \tau_2}{E \vdash e_1\ e_2 : \tau_1} \text{ (app)}$$

$$\frac{E, x : \tau_1 \vdash e : \tau_2}{E \vdash \lambda x.e : \tau_1 \to \tau_2} \text{ (abs)} \qquad \frac{E \vdash e_1 : \tau_1 \quad E, x : \tau_1 \vdash e_2 : \tau_2}{E \vdash \textbf{let}\ x = e_1\ \textbf{in}\ e_2 : \tau_2} \text{ (let)}$$

$$\frac{E \vdash e_1 : \text{bool} \quad E \vdash e_2 : \tau \quad E \vdash e_3 : \tau}{E \vdash \textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3 : \tau} \text{ (ite)}$$

- environment $E = \{\text{true} : \text{bool}, + : \text{int} \to \text{int} \to \text{int}\}$
- judgment $E \vdash (\lambda x.x)\ \text{true} : \text{bool}$

Proof.

$$\dfrac{\dfrac{E, x : \text{bool} \vdash x : \text{bool}}{E \vdash \lambda x.x : \text{bool} \to \text{bool}}\ (\text{abs}) \qquad E \vdash \text{true} : \text{bool}}{E \vdash (\lambda x.x)\ \text{true} : \text{bool}}\ (\text{app})$$

□

- environment $E = \{\text{true} : \text{bool}, + : \text{int} \to \text{int} \to \text{int}\}$
- judgment $E \vdash \lambda x.x + x : \text{int} \to \text{int}$

Try it!

What is Type Inference?

- Given some environment
- together with a core ML expression
- and a type
- infer a unifier (type substitution) —if possible—
- such that the most general type of the expression is obtained

Preliminaries (full functional notation)

Type variables:

$$\mathcal{TV}\mathrm{ar}(\tau) \stackrel{\text{def}}{=} \begin{cases} \{\alpha\} & \text{if } \tau = \alpha \\ \mathcal{TV}\mathrm{ar}(\tau_1) \cup \mathcal{TV}\mathrm{ar}(\tau_2) & \text{if } \tau = \tau_1 \to \tau_2 \\ \bigcup_{1 \le i \le n} \mathcal{TV}\mathrm{ar}(\tau_i) & \text{if } \tau = g(\tau_1, \ldots, \tau_n) \end{cases}$$

Type substitution: σ is mapping from type variables to types

Application:

$$\tau\sigma \stackrel{\text{def}}{=} \begin{cases} \sigma(\alpha) & \text{if } \tau = \alpha \\ \tau_1\sigma \to \tau_2\sigma & \text{if } \tau = \tau_1 \to \tau_2 \\ g(\tau_1\sigma, \ldots, \tau_n\sigma) & \text{if } \tau = g(\tau_1, \ldots, \tau_n) \end{cases}$$

$$E\sigma \stackrel{\text{def}}{=} \{e : \tau\sigma \mid e : \tau \in E\}$$

Composition: $\sigma_1\sigma_2 \stackrel{\text{def}}{=} \sigma_2 \circ \sigma_1$, i.e., $\alpha \mapsto \sigma_2(\sigma_1(\alpha))$

Example

$$\tau = \alpha \to (\alpha_1 \to \alpha_3)$$

$$\sigma = \{\alpha/\mathsf{int} \to \mathsf{int}, \alpha_1/\mathsf{list}(\alpha_2)\}$$

$$\sigma_2 = \{\alpha_3/\alpha_4, \alpha_2/\alpha, \alpha/\alpha_1\}$$

$$\mathcal{TV}\mathsf{ar}(\tau) = \{\alpha, \alpha_1, \alpha_3\}$$

$$\tau\sigma = (\mathsf{int} \to \mathsf{int}) \to (\mathsf{list}(\alpha_2) \to \alpha_3)$$

$$\mathcal{TV}\mathsf{ar}(\tau\sigma) = \{\alpha_2, \alpha_3\}$$

$$\sigma\sigma_2 = \{\alpha/\mathsf{int} \to \mathsf{int}, \alpha_1/\mathsf{list}(\alpha), \alpha_3/\alpha_4, \alpha_2/\alpha\}$$

Unification Problems

Definition

- unification problem is (finite) sequence of equations

$$\tau_1 \approx \tau_1'; \ldots ; \tau_n \approx \tau_n'$$

- $\square$ denotes empty sequence
- type substitution σ is unifier of unification problem if

$$\tau_1\sigma = \tau_1'\sigma; \ldots ; \tau_n\sigma = \tau_n'\sigma$$

- process of computing a unifier is called unification

The Unification System $\mathcal{U}$

$$\frac{E_1; g(\tau_1, \ldots, \tau_n) \approx g(\tau_1', \ldots, \tau_n'); E_2}{E_1; \tau_1 \approx \tau_1'; \ldots; \tau_n \approx \tau_n'; E_2}\ (\mathrm{d}_1)$$

$$\frac{E_1; \tau_1 \to \tau_2 \approx \tau_1' \to \tau_2'; E_2}{E_1; \tau_1 \approx \tau_1'; \tau_2 \approx \tau_2'; E_2}\ (\mathrm{d}_2)$$

$$\frac{E_1; \alpha \approx \tau; E_2 \quad \alpha \notin \mathcal{TV}\mathrm{ar}(\tau)}{(E_1; E_2)\{\alpha/\tau\}}\ (\mathrm{v}_1)$$

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$$\frac{E_1; \tau \approx \tau; E_2}{E_1; E_2}\ (\mathrm{t})$$

Unification Problem (cont'd)

Notation

$E \Rightarrow_{\sigma}^{(r)} E'$ if rule r from $\mathcal{U}$ applied to equations E yields E'

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Theorem

if $E_1 \Rightarrow_{\sigma_1}^{(r_1)} E_2 \Rightarrow_{\sigma_2}^{(r_2)} \dots \Rightarrow_{\sigma_{n-1}}^{(r_{n-1})} \square$ *then* E_1 *has unifier* $\sigma_1 \cdots \sigma_{n-1}$

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Example

$$\begin{aligned} \mathsf{list}(\mathsf{bool}) \approx \mathsf{list}(\alpha) \quad & \Rightarrow_{\iota}^{(\mathsf{d}_1)} \quad \mathsf{bool} \approx \alpha \\ & \Rightarrow_{\{\alpha/\mathsf{bool}\}}^{(\mathsf{v}_2)} \quad \square \end{aligned}$$

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$E \Rightarrow_{\sigma}^{(r)} E'$ if rule r from $\mathcal{U}$ applied to equations E yields E'

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Remarks

- unification always terminates
- the order of applying inference rules has no (dramatic) effect

Type Inference Problems

- $E \triangleright e : \alpha_0$ is type inference problem
- σ s.t., $E\sigma \vdash e : \alpha_0\sigma$ (if exists) is solution (otherwise: e not typable)

The Type Inference System $\mathcal{I}$

$$\frac{E, e : \tau_0 \rhd e : \tau_1}{\tau_0 \approx \tau_1} \text{ (con)} \qquad \frac{E \rhd e_1\ e_2 : \tau}{E \rhd e_1 : \alpha \to \tau; E \rhd e_2 : \alpha} \text{ (app)}$$

$$\frac{E \rhd \lambda x.e : \tau}{E, x : \alpha_1 \rhd e : \alpha_2; \tau \approx \alpha_1 \to \alpha_2} \text{ (abs)} \qquad \frac{E \rhd \textbf{let } x = e_1 \textbf{ in } e_2 : \tau}{E \rhd e_1 : \alpha; E, x : \alpha \rhd e_2 : \tau} \text{ (let)}$$

$$\frac{E \rhd \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3 : \tau}{E \rhd e_1 : \text{bool}; E \rhd e_2 : \tau; E \rhd e_3 : \tau} \text{ (ite)}$$

Recipe - Type Inference

Input
core ML expression e and typing environment E

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core ML expression e and typing environment E

Algorithm

1. start with $E \triangleright e : \alpha_0$ (fresh type variable α_0)
2. use $\mathcal{I}$ to transform $E \triangleright e : \alpha_0$ into unification problem u
   (if at any point no rule applicable Not Typable)
3. if possible solve u (obtaining unifier σ) otherwise Not Typable

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core ML expression e and typing environment E

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Output

the most general type of e w.r.t. E is $\alpha_0\sigma$

Example

Find most general type of **let** $id = \lambda x.x$ **in** id 1 w.r.t. P

Some notes on unification:

The algorithms for first-order unification are worst-case exponential (Robinson’s algorithm).

There exist linear algorithms, but with a really bad constant, so exponential ones are used in practice

Unification for more complex structures quickly becomes hard, in fact higher-order unification is undecidable

Additional Literature (not required)

More on efficient unification.

Krystof Hoder and Andrei Voronkov.
Comparing unification algorithms in first-order theorem proving.
In Bärbel Mertsching, Marcus Hund, and Muhammad Zaheer Aziz, editors, *KI 2009: Advances in Artificial Intelligence, 32nd Annual German Conference on AI*, volume 5803 of *LNCS*, pages 435–443. Springer, 2009.

Work Here / Homework

Try to find the most general type, if possible

- λf x. x(f x)
- λf. (λx. f (x x)) (λx. f (x x))