universität innsbruck

WS 2024 lecture 1

Automata and Logic

Aart Middeldorp and Johannes Niederhauser

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  1. Automata and Logic
  2. Constraint Solving
  3. Cryptography
  4. High−Performance Computing
  5. Optimisation and Numerical Computation
  6. Signal Processing and Algorithmic Geometry

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  - Tree Automata (WM 9)

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  - Program and Resource Analysis (WM 8)
  - Tree Automata (WM 9)
  - Semantics of Programming Languages (WM 7)
  - Quantum Computing (WM 8)
  - Research Seminar (WM 9)

Outline

1. **Introduction**

   Organisation Contents

2. **Basic Definitions**

3. **Deterministic Finite Automata**

4. **Intermezzo**

5. **Closure Properties**

6. **Further Reading**

VO is streamed and recorded

articify with session ID **8020 8256** for anonymous questions

Important Information

- LVA 703302 (VO 2) + 703303 (PS 2)

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Time and Place

VO	Monday	8:15 – 10:00	HSB 9	(AM)
PS	Friday	8:15 – 10:00	SR 12	(JN)

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Consultation Hours

Aart Middeldorp	3M07	Wednesday	11:30 – 13:00
Johannes Niederhauser	3M03	Thursday	9:00 – 10:30

Schedule

week 1	07.10 & 11.10	week 6	11.11 & 15.11	week 11	16.12 & 10.01
week 2	14.10	week 7	18.11 & 22.11	week 12	13.01 & 17.01
week 3	21.10 & 25.10	week 8	25.11 & 29.11	week 13	20.01 & 24.01
week 4	28.10	week 9	02.12 & 06.12	week 14	27.01
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Grading – Vorlesung

- first exam on January 27

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- second exam on February 26
- third exam on September 25 (on demand)

Grading – PS

$$\text{score} \;=\; \min\left(\tfrac{10}{13}(E+P)+B, 100\right)$$

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grade : $[0,50) \to \mathbf{5}$ $\quad [50,63) \to \mathbf{4}$ $\quad [63,75) \to \mathbf{3}$ $\quad [75,88) \to \mathbf{2}$ $\quad [88,100] \to \mathbf{1}$

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evaluation 2023W

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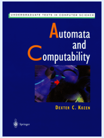

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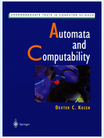

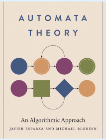

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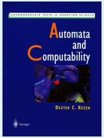

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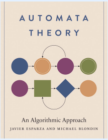

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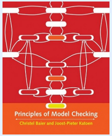

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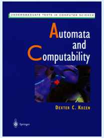

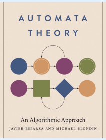

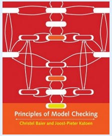

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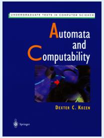

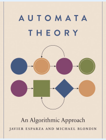

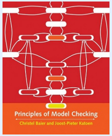

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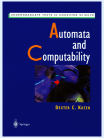

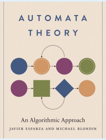

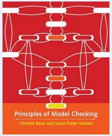

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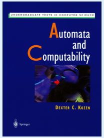

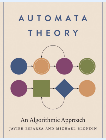

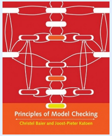

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3. Deterministic Finite Automata

4. Intermezzo

5. Closure Properties

6. Further Reading

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata
- tree automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

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Examples

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- $\{x \mid x$ is valid program in some machine language$\}$

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- $\#a(x)$ ($a \in \Sigma$, $x \in \Sigma^*$) denotes number of a's in x

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Examples

- substrings of 011: $0, 1, 01, 11, 011$

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- $\{1, 01\}^* = \{\epsilon, 1, 01, 11, 011, 101, 0101, 111, 0111, 1011, 01011, \ldots\}$
- $2^{\{1,01\}} = \{\varnothing, \{1\}, \{01\}, \{1, 01\}\}$

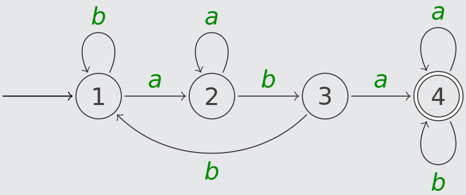
Some Useful Properties

- $\{\epsilon\}A = A\{\epsilon\} = A$
- $\varnothing A = A\varnothing = \varnothing$
- $\sim(A \cup B) = (\sim A) \cap (\sim B)$
- $\sim(A \cap B) = (\sim A) \cup (\sim B)$
- $A^{m+n} = A^m A^n$
- $A^* A^* = A^*$
- $A^{**} = A^*$
- $A^* = \{\epsilon\} \cup AA^* = \{\epsilon\} \cup A^* A$
- $\varnothing^* = \{\epsilon\}$

Outline

1. Introduction
2. Basic Definitions
3. **Deterministic Finite Automata**
4. Intermezzo
5. Closure Properties
6. Further Reading

Example



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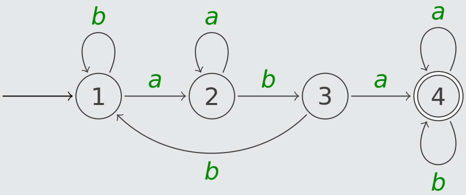

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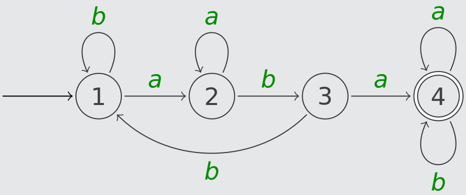

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  ④ $s \in Q$: start state

  ⑤ $F \subseteq Q$: final (accept) states

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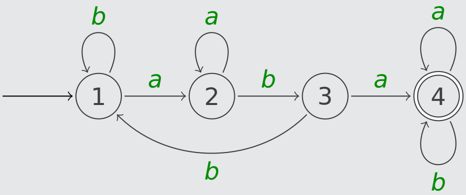

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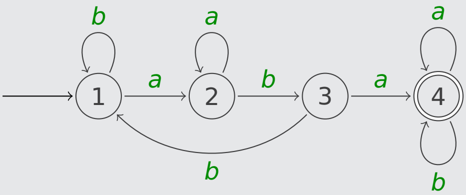

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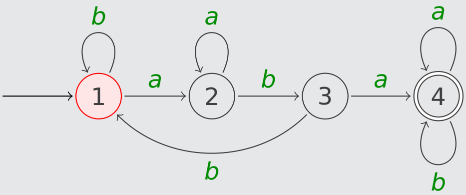

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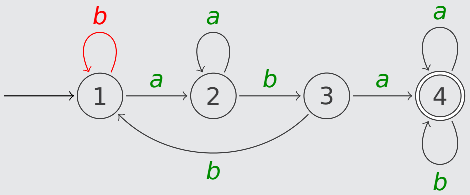

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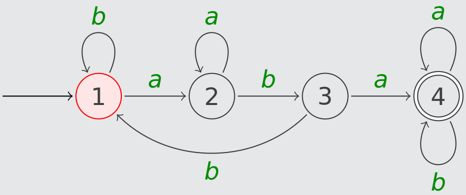

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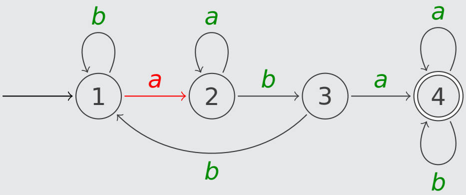

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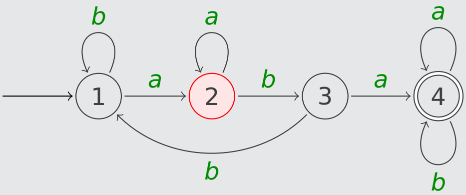

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	b	a	b	a	a
1	1	2			

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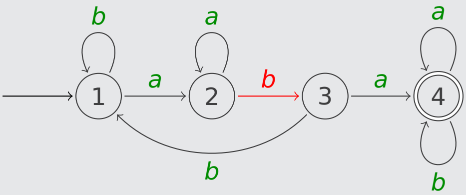

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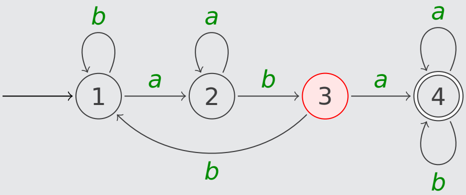

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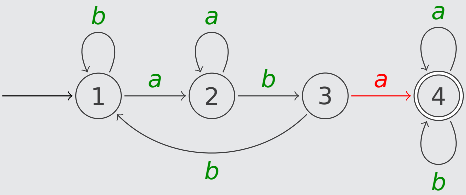

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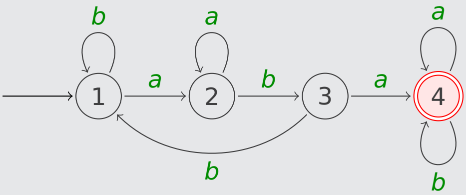

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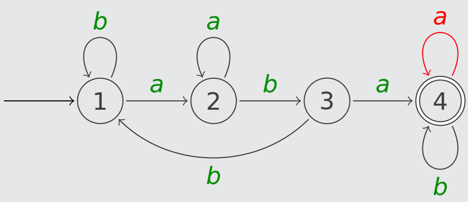

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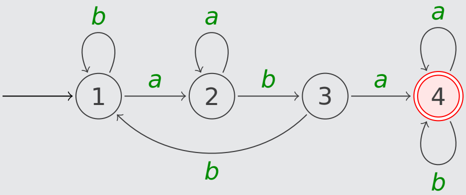

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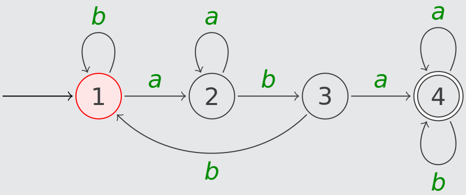

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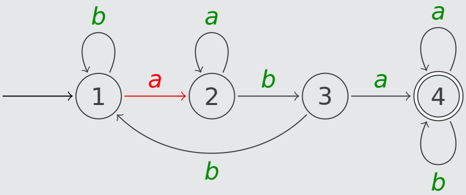
Example

DFA $M = (Q, \Sigma, \delta, s, F)$



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δ	a	b
1	2	1
2	2	3
3	4	1
4	4	4

$b \quad a \quad b \quad a \quad a$

$1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 4$

$a \quad a \quad b \quad b \quad b$

1

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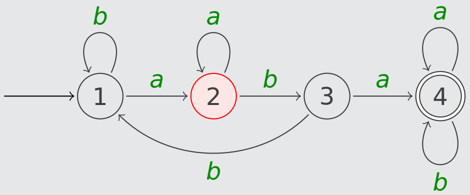

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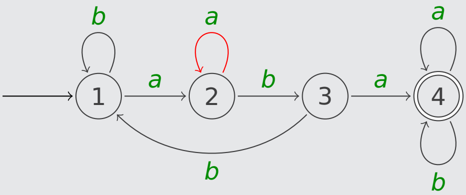

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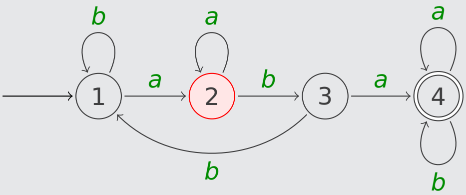

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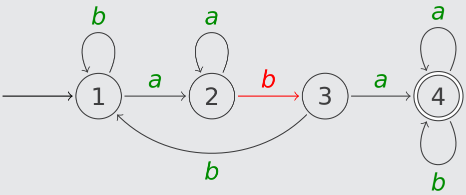

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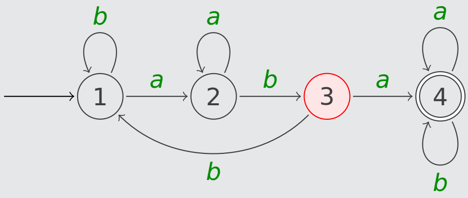

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$1 \quad 2 \quad 2 \quad 3$

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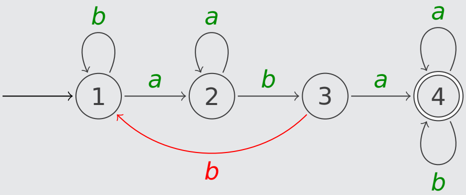

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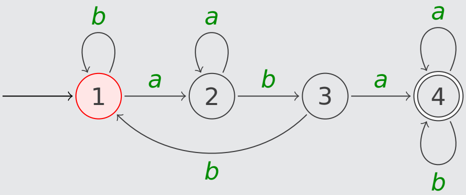

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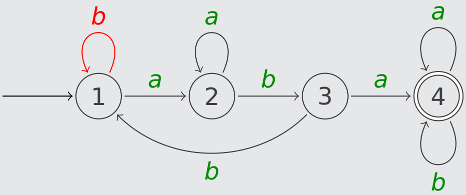

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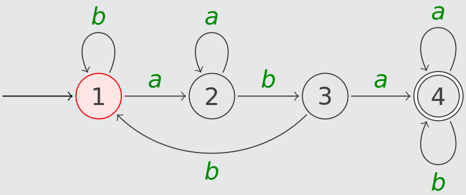

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	a	a	b	b	b
1	2	2	3	1	1

Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\delta \colon Q \times \Sigma \to Q$: transition function
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  5. $F \subseteq Q$: final (accept) states
- $\widehat{\delta} \colon Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \epsilon) = q \qquad \widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$$

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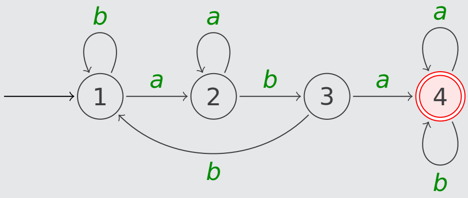
Example

DFA $M = (Q, \Sigma, \delta, s, F)$



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	b	a	b	a	a	$\in L(M)$
1	1	2	3	4	4	

	a	a	b	b	b
1	2	2	3	1	1

Example

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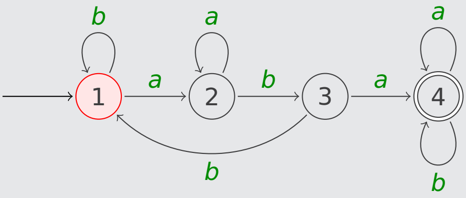

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1 1 2 3 4 4

$a \ a \ b \ b \ b \notin L(M)$

1 2 2 3 1 1

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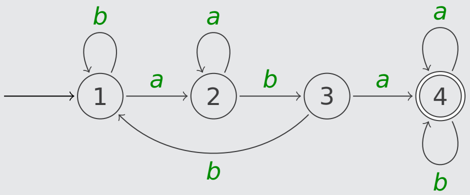

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$$L(M) = \{x \mid \qquad\qquad \}$$

Example

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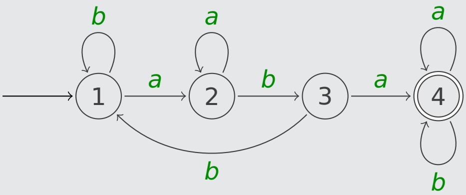

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$$L(M) = \{x \mid x \text{ contains } aba \text{ as substring}\}$$

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- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M: $L(M) = \{x \mid \widehat{\delta}(s, x) \in F\}$
- set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Outline

1. Introduction
2. Basic Definitions
3. Deterministic Finite Automata
4. **Intermezzo**
5. Closure Properties
6. Further Reading

Question

What is the language accepted by the DFA given by the following transition table ?

		a	b
$\rightarrow$	1	1	2
	2 F	2	3
	3	3	3

Here the arrow indicates the start state and F marks the final states.

A $\{a^n b \mid n \in \mathbb{N}\}$

B $\sim(\{a\}^*\{b\}\{a,b\}^*)$

C the set of strings over $\{a,b\}$ containing exactly one b

D the set of strings over $\{a,b\}$ that do not contain two or more b's

Outline

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Theorem

regular sets are effectively closed under intersection

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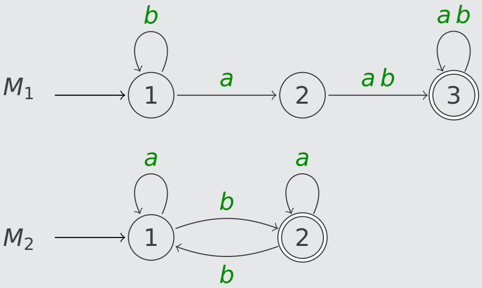
Example



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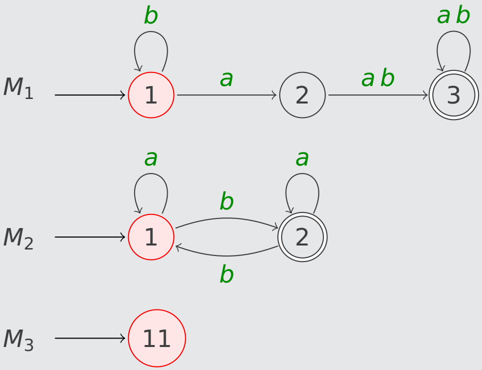

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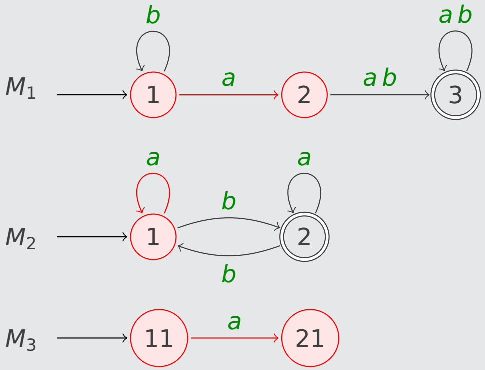

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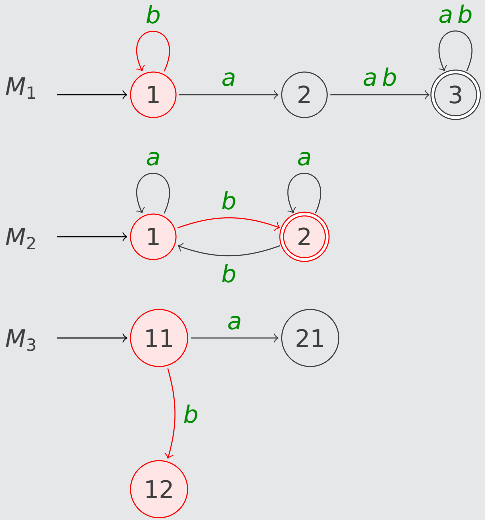

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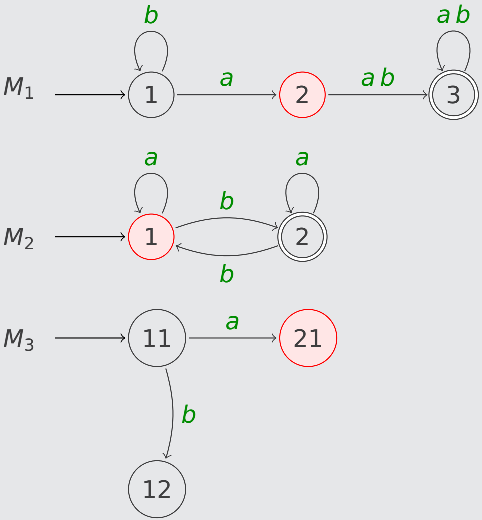

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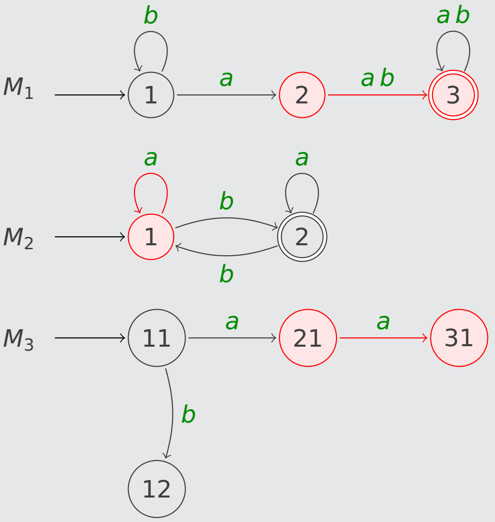

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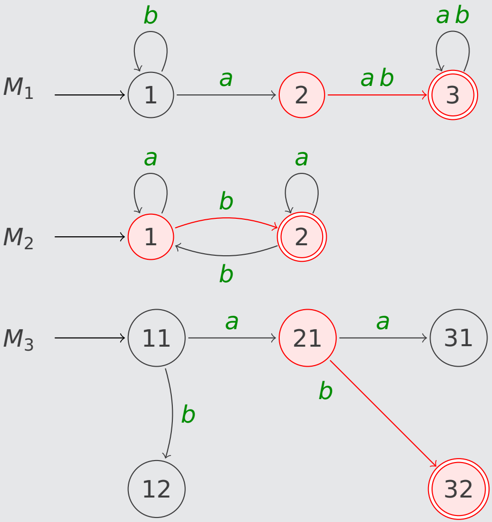

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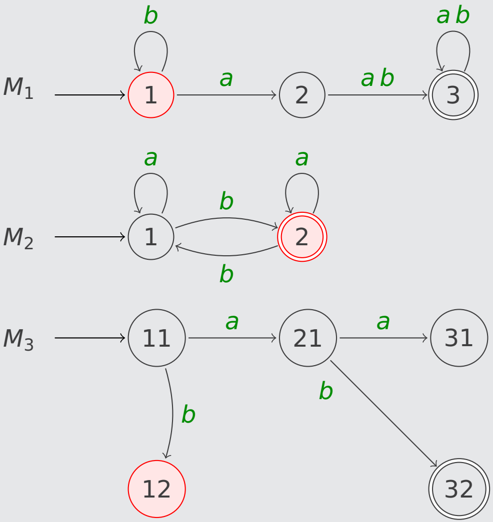

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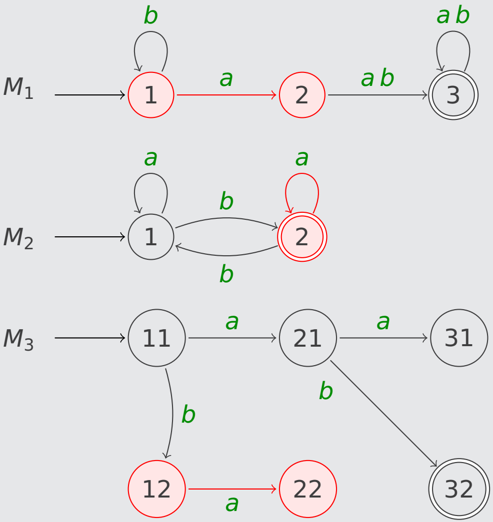

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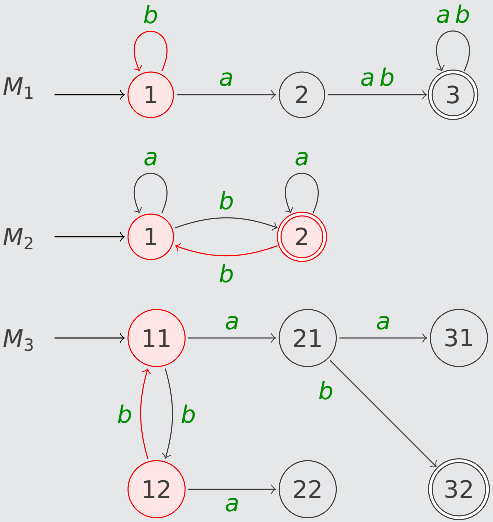

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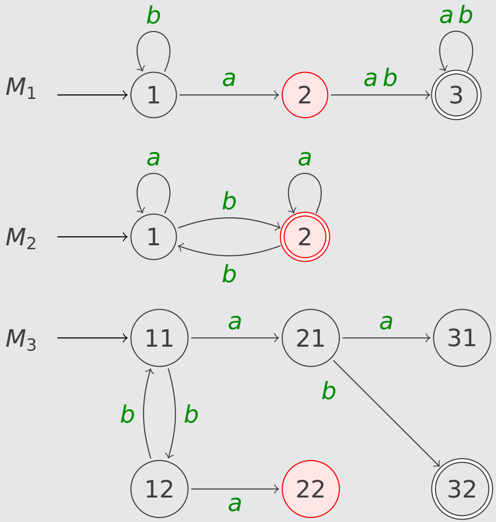

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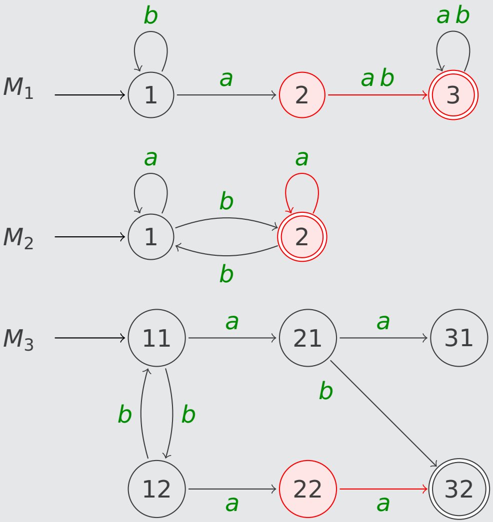

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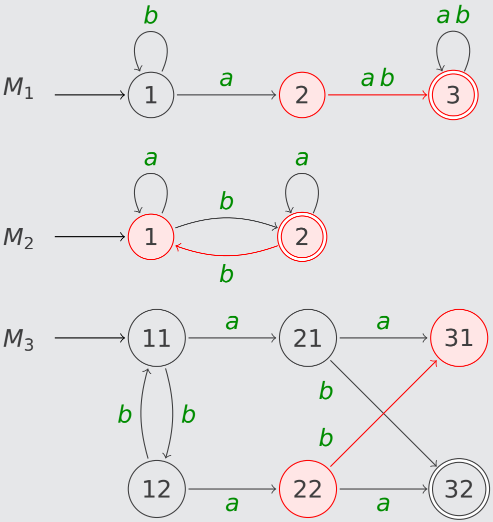

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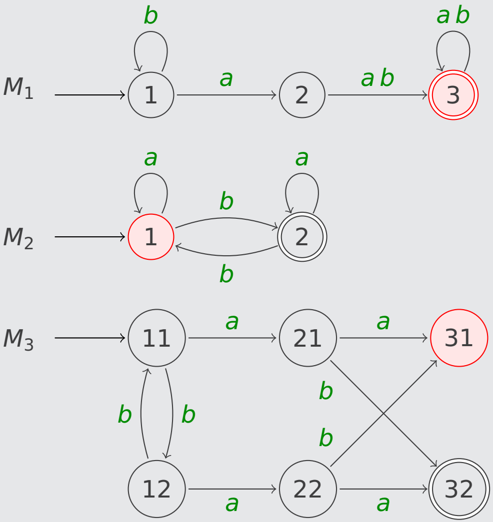

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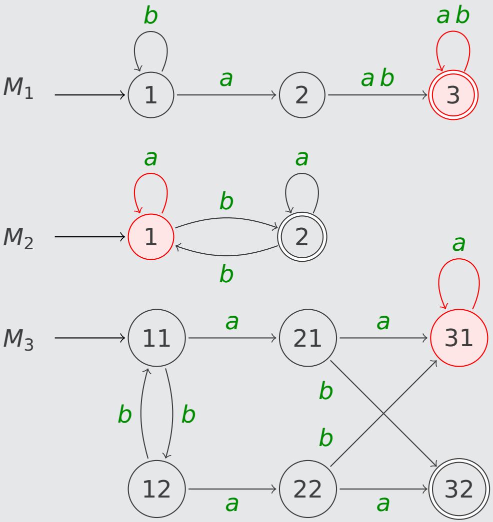

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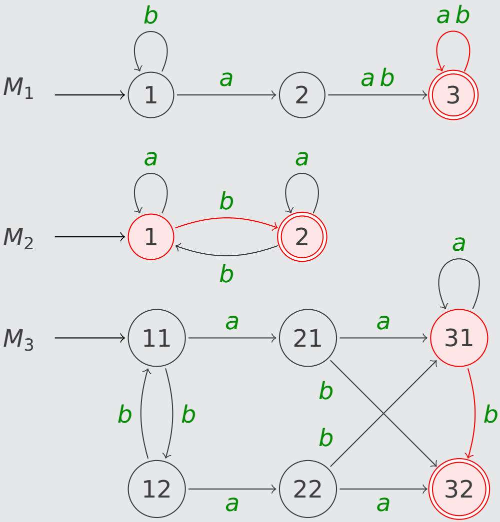

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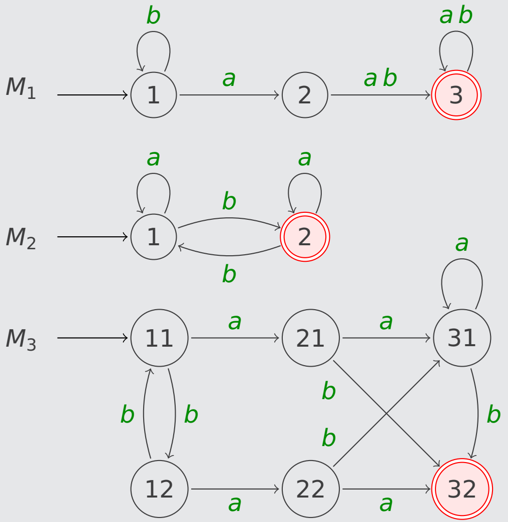

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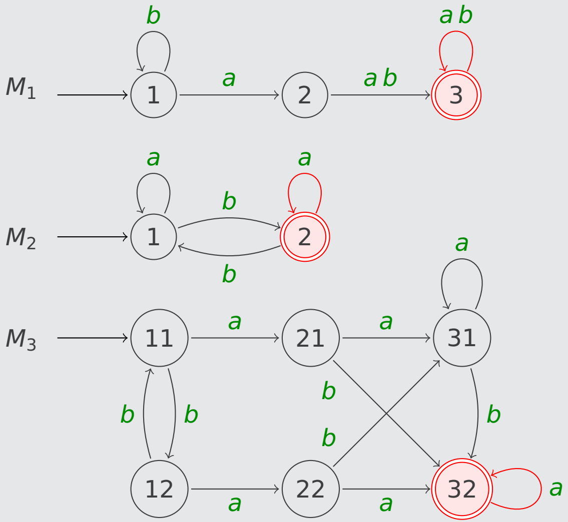

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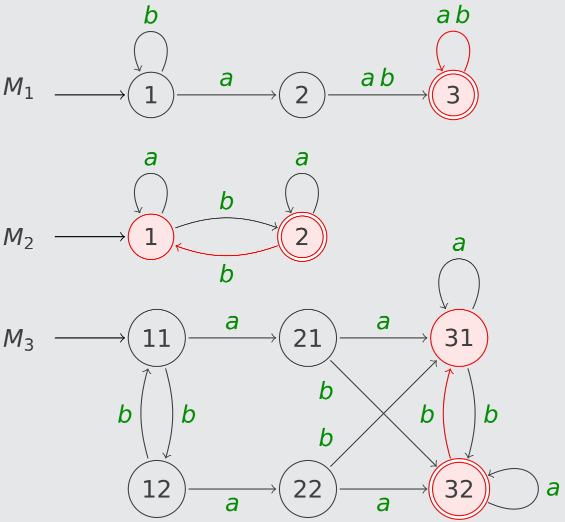

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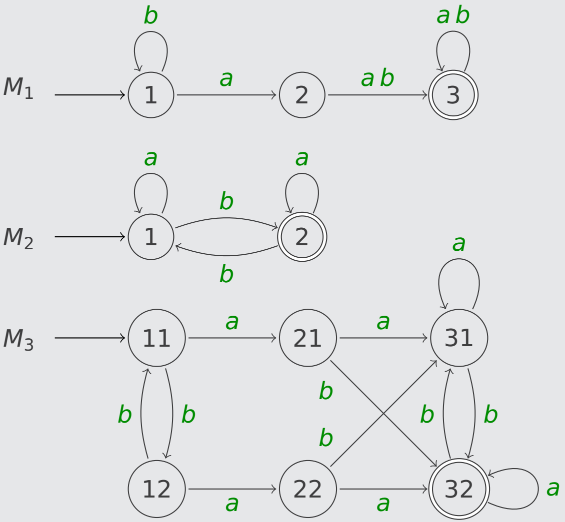

$$L(M_3) = L(M_1) \cap L(M_2)$$

Theorem

regular sets are effectively closed under intersection

Proof

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

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  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
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- $A \cap B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with

  ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$

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- $A \cap B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with

  ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$

  ② $F_3 = F_1 \times F_2$

Theorem

regular sets are effectively closed under intersection

Proof

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  ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for all $p \in Q_1$, $q \in Q_2$, $a \in \Sigma$

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- claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x))$ for all $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$ (on next slide)

Theorem

regular sets are effectively closed under intersection

Proof (product construction)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- $A \cap B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with
  1. $Q_3 = Q_1 \times Q_2 = \{(p,q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
  2. $F_3 = F_1 \times F_2$
  3. $s_3 = (s_1, s_2)$
  4. $\delta_3((p,q),a) = (\delta_1(p,a), \delta_2(q,a))$ for all $p \in Q_1$, $q \in Q_2$, $a \in \Sigma$
- claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x))$ for all $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$ (on next slide)

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

- base case: $|x| = 0$

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

- base case: $|x| = 0$ and thus $x = \epsilon$

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

- base case: $|x| = 0$ and thus $x = \epsilon$

$$\widehat{\delta_3}((p,q),x) = (p,q) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x))$$

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- induction step: $|x| > 0$

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

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- induction step: $|x| > 0$ and thus $x = ya$ with $|y| = |x| - 1$

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$$\widehat{\delta_3}((p,q),x) = \delta_3(\widehat{\delta_3}((p,q),y),a) \qquad \text{(definition of } \widehat{\delta_3}\text{)}$$

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x),\widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

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$$\begin{aligned}\widehat{\delta_3}((p,q),x) &= \delta_3(\widehat{\delta_3}((p,q),y),a) && \text{(definition of } \widehat{\delta_3}\text{)}\\ &= \delta_3((\widehat{\delta_1}(p,y),\widehat{\delta_2}(q,y)),a) && \text{(induction hypothesis)}\end{aligned}$$

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

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Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

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$$\begin{aligned}
\widehat{\delta_3}((p,q),x) &= \delta_3(\widehat{\delta_3}((p,q),y),a) && \text{(definition of } \widehat{\delta_3}\text{)} \\
&= \delta_3((\widehat{\delta_1}(p,y), \widehat{\delta_2}(q,y)),a) && \text{(induction hypothesis)} \\
&= (\delta_1(\widehat{\delta_1}(p,y),a), \delta_2(\widehat{\delta_2}(q,y),a)) && \text{(definition of } \delta_3\text{)} \\
&= (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) && \text{(definition of } \widehat{\delta_1} \text{ and } \widehat{\delta_2}\text{)}
\end{aligned}$$

Theorem

regular sets are effectively closed under complement

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Proof

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

Theorem

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Proof

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
- $\sim A = \Sigma^* - A = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ with

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  ① $Q_2 = Q_1$

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  1. $Q_2 = Q_1$
  2. $\delta_2(q, a) = \delta_1(q, a) \quad \forall\, q \in Q_2 \,\forall\, a \in \Sigma$
  3. $s_2 = s_1$

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- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
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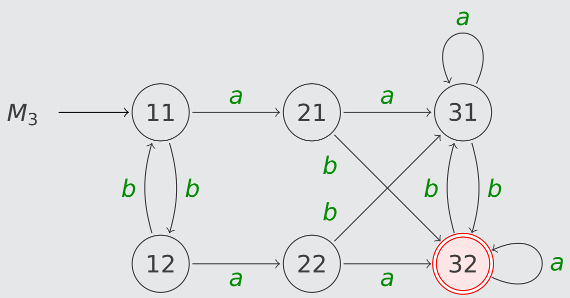
  ② $\delta_2(q, a) = \delta_1(q, a) \quad \forall\, q \in Q_2\ \forall\, a \in \Sigma$

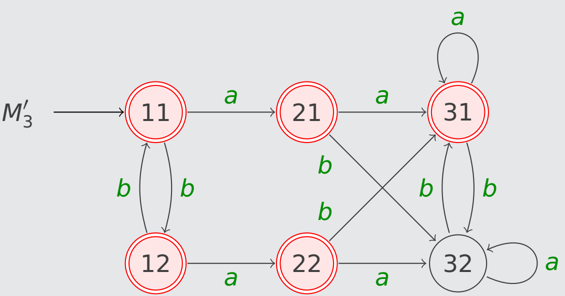
  ③ $s_2 = s_1$

  ④ $F_2 = Q_1 - F_1$

Example



Example



$$L(M_3') = \sim L(M_3)$$

Theorem

regular sets are effectively closed under union

Theorem

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Proof

$A \cup B = \sim((\sim A) \cap (\sim B))$

Theorem

regular sets are effectively closed under union

Proof (explicit construction)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

- $A \cup B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with

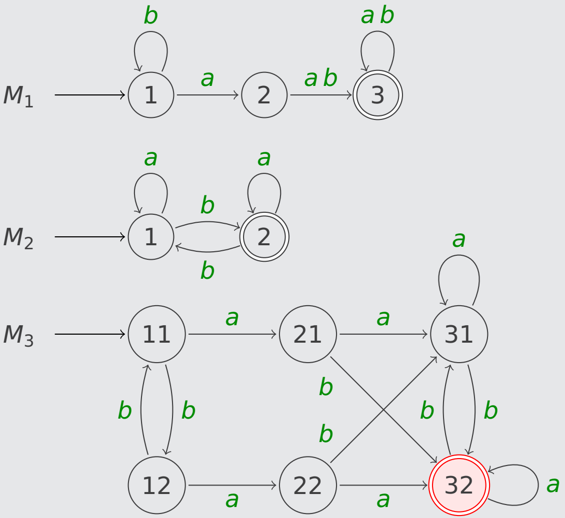
  ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$

  ② $F_3 = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

  ③ $s_3 = (s_1, s_2)$

  ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a)) \quad \forall p \in Q_1\ \forall q \in Q_2\ \forall a \in \Sigma$

Example



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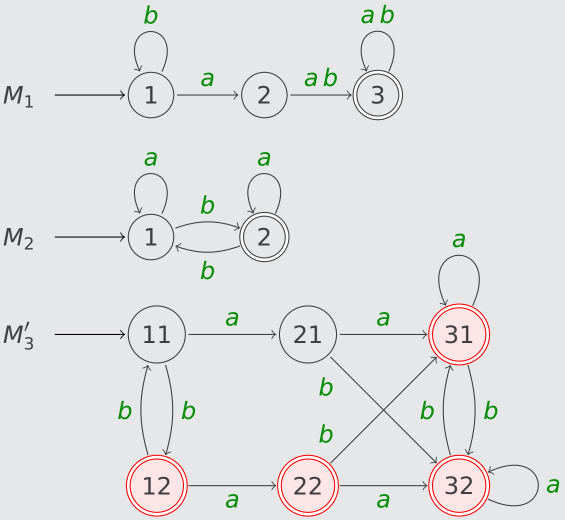

$$L(M_3') = L(M_1) \cup L(M_2)$$

Outline

1. Introduction
2. Basic Definitions
3. Deterministic Finite Automata
4. Intermezzo
5. Closure Properties
6. **Further Reading**

Kozen

- Lectures 1 – 4

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Important Concepts

- alphabet
- language
- regular set
- closure properties
- product construction
- string
- DFA

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homework for October 11

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Solutions

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