WS 2024 lecture 1

Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Initial Remarks

- Automata and Logic is elective module 1 in master program Computer Science
- master students must select 3 out of 6 elective modules:
  1. Automata and Logic
  2. Constraint Solving (offered in 2025S)
  3. Cryptography
  4. High−Performance Computing
  5. Optimisation and Numerical Computation
  6. Signal Processing and Algorithmic Geometry
- other master modules with theory content (Logic and Learning specialization):
  - Program and Resource Analysis (WM 8)
  - Tree Automata (WM 9)
  - Semantics of Programming Languages (WM 7)
  - Quantum Computing (WM 8)
  - Research Seminar (WM 9)

Outline

VO is streamed and recorded

Particify with session ID **8020 8256** for anonymous questions

Important Information

- LVA 703302 (VO 2) + 703303 (PS 2)
- http://cl-informatik.uibk.ac.at/teaching/ws24/al
- online registration for VO required
- OLAT links for VO and PS

Time and Place

VO	Monday	8:15 – 10:00	HSB 9	(AM)
PS	Friday	8:15 – 10:00	SR 12	(JN)

Consultation Hours

Aart Middeldorp	3M07	Wednesday	11:30 – 13:00
Johannes Niederhauser	3M03	Thursday	9:00 – 10:30

Schedule

week 1	07.10 & 11.10	week 6	11.11 & 15.11	week 11	16.12 & 10.01
week 2	14.10	week 7	18.11 & 22.11	week 12	13.01 & 17.01
week 3	21.10 & 25.10	week 8	25.11 & 29.11	week 13	20.01 & 24.01
week 4	28.10	week 9	02.12 & 06.12	week 14	27.01 (first exam)
week 5	04.11 & 08.11	week 10	09.12 & 13.12		

Grading – Vorlesung

- first exam on January 27
- registration starts 5 weeks and ends 2 weeks before exam
- de-registration is possible until 10:00 on January 24
- second exam on February 26
- third exam on September 25 (on demand)

Grading – PS

$$\text{score} = \min\left(\tfrac{10}{13}(E+P)+B, 100\right)$$

E: points for solved exercises (at most 110)

B: points for bonus exercises (at most 20)

P: points for presentations of solutions (at most 20)

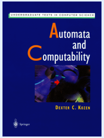
grade: $[0,50) \to \mathbf{5}$ $[50,63) \to \mathbf{4}$ $[63,75) \to \mathbf{3}$ $[75,88) \to \mathbf{2}$ $[88,100] \to \mathbf{1}$

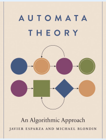
- homework exercises are given on course web site
- solved exercises must be marked and solutions must be uploaded (PDF) in OLAT
- strict deadline: 7 am on Friday
- 10 points per PS
- two presentations of solutions are mandatory
- 20 points for two presentations; additional presentations give bonus points
- attendance is compulsory; unexcused absence is allowed twice (resulting in 0 points)

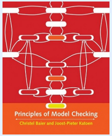
evaluation 2023W

Literature

- Dexter C Kozen
  Automata and Computability
  Springer–Verlag, 1997
- Javier Esparza and Michael Blondin
  Automata Theory: An Algorithmic Approach
  MIT Press, 2023
- Christel Baier and Joost–Pieter Katoen
  Principles of Model Checking
  MIT Press, 2008
- additional resources will be linked from course website







Online Material

- ~~access to slides and exercises is restricted to uibk.ac.at domain~~
- solutions to selected exercises are available after they have been discussed in PS

Outline

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata
- tree automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Definitions

- alphabet is finite set; its elements are called symbols or letters
- string over alphabet Σ is finite sequence of elements of Σ
- length $|x|$ of string x is number of symbols in x
- empty string is unique string of length 0 and denoted by ϵ
- Σ^* is set of all strings over Σ $\quad (\varnothing^* = \{\epsilon\})$
- language over Σ is subset of Σ^*

Examples

strings over $\Sigma = \{0, 1\}$: $\quad 0 \quad 0110 \quad \epsilon$

languages over Σ:

- $\{\epsilon, 0, 1, 00, 01, 10, 11\}$ $\quad$ (all strings having at most two symbols)
- $\{x \mid x$ is valid program in some machine language$\}$

Definitions

- string concatenation $x, y \in \Sigma^* \implies xy \in \Sigma^*$ is associative:

$$(xy)z = x(yz) \qquad \text{for all } x, y, z \in \Sigma^*$$

- empty string is identity for concatenation:

$$\epsilon x = x\epsilon = x \qquad \text{for all } x \in \Sigma^*$$

- x is substring (prefix, suffix) of y if $y = uxv$ $(y = xv,\ y = ux)$

- x^n $(x \in \Sigma^*,\ n \in \mathbb{N})$:

$$x^0 = \epsilon$$
$$x^{n+1} = x^n x$$

- $\#a(x)$ $(a \in \Sigma,\ x \in \Sigma^*)$ denotes number of a's in x

Definitions

for $A, B \subseteq \Sigma^*$

- union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- complement $\sim A = \Sigma^* - A = \{x \in \Sigma^* \mid x \notin A\}$
- set concatenation $AB = \{xy \mid x \in A \text{ and } y \in B\}$
- powers A^n $(n \in \mathbb{N})$ $A^0 = \{\epsilon\}$ $A^{n+1} = AA^n$
- asterate A^* is union of all finite powers of A

$$A^* = \bigcup_{n \geqslant 0} A^n = \{x_1 x_2 \cdots x_n \mid n \geqslant 0 \text{ and } x_i \in A \text{ for all } 1 \leqslant i \leqslant n\}$$

- $A^+ = AA^* = \bigcup_{n \geqslant 1} A^n$
- power set $2^A = \{Q \mid Q \subseteq A\}$

Examples

- substrings of 011: 0, 1, 01, 11, 011, ϵ
- prefixes of 011: 0, 01, 011, ϵ
- suffixes of 011: 1, 11, 011, ϵ
- $(011)^3 = 011011011 \neq 011^3$
- $\#1(011011011) = 6$ $\quad\#0(\epsilon) = 0$
- $\{0, 10, 111\}\{1, 11\} = \{01, 101, 1111, 011, 1011, 11111\}$
- $\{0, 01, 111\}\{1, 11\} = \{01, 011, 1111, 0111, 11111\}$
- $\{1, 01\}^3 = \{111, 0111, 1011, 01011, 1101, 01101, 10101, 010101\}$
- $\{1, 01\}^* = \{\epsilon, 1, 01, 11, 011, 101, 0101, 111, 0111, 1011, 01011, \ldots\}$
- $2^{\{1,01\}} = \{\varnothing, \{1\}, \{01\}, \{1, 01\}\}$

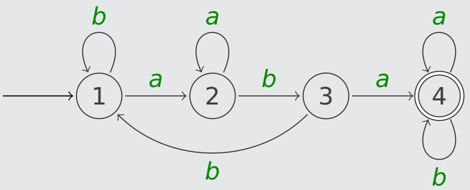
Some Useful Properties

- $\{\epsilon\}A = A\{\epsilon\} = A$
- $\varnothing A = A\varnothing = \varnothing$
- $\sim(A \cup B) = (\sim A) \cap (\sim B)$
- $\sim(A \cap B) = (\sim A) \cup (\sim B)$
- $A^{m+n} = A^m A^n$
- $A^* A^* = A^*$
- $A^{**} = A^*$
- $A^* = \{\epsilon\} \cup AA^* = \{\epsilon\} \cup A^* A$
- $\varnothing^* = \{\epsilon\}$

Outline

Example

DFA $M = (Q, \Sigma, \delta, s, F)$



1. $Q = \{1, 2, 3, 4\}$
2. $\Sigma = \{a, b\}$
3. $\delta \colon Q \times \Sigma \to Q$
4. $s = 1$
5. $F = \{4\}$

δ	a	b
1	2	1
2	2	3
3	4	1
4	4	4

$\;\; b \;\; a \;\; b \;\; a \;\; a \;\; \in L(M)$
$1 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 4$

$\;\; a \;\; a \;\; b \;\; b \;\; b \;\; \notin L(M)$
$1 \;\; 2 \;\; 2 \;\; 3 \;\; 1 \;\; 1$

$L(M) = \{x \mid x \text{ contains } aba \text{ as substring}\}$

Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\delta \colon Q \times \Sigma \to Q$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\delta} \colon Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \epsilon) = q \qquad \widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is accepted by M if $\widehat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M: $L(M) = \{x \mid \widehat{\delta}(s, x) \in F\}$
- set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Outline

Question

What is the language accepted by the DFA given by the following transition table ?

		a	b
$\rightarrow$	1	1	2
	2 F	2	3
	3	3	3

Here the arrow indicates the start state and F marks the final states.

A $\{a^n b \mid n \in \mathbb{N}\}$

B $\sim(\{a\}^*\{b\}\{a,b\}^*)$

C the set of strings over $\{a,b\}$ containing exactly one b

D the set of strings over $\{a,b\}$ that do not contain two or more b's

Outline

Theorem

regular sets are effectively closed under intersection

Proof (product construction)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- $A \cap B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with

  ① $Q_3 = Q_1 \times Q_2 = \{(p,q) \mid p \in Q_1 \text{ and } q \in Q_2\}$

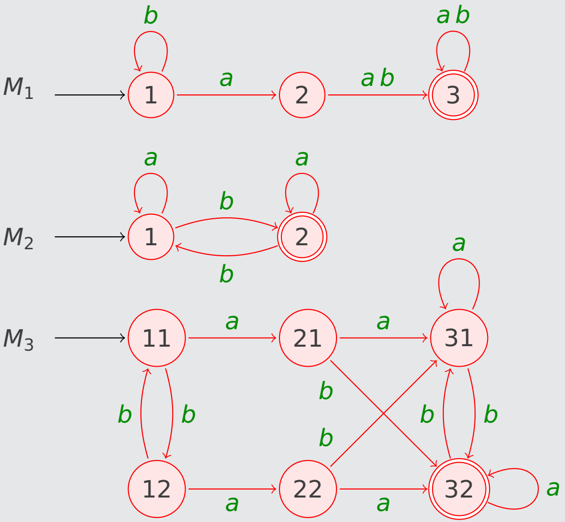
  ② $F_3 = F_1 \times F_2$

  ③ $s_3 = (s_1, s_2)$

  ④ $\delta_3((p,q),a) = (\delta_1(p,a), \delta_2(q,a))$ for all $p \in Q_1$, $q \in Q_2$, $a \in \Sigma$
- claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x))$ for all $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$ (on next slide)

Example



$$L(M_3) = L(M_1) \cap L(M_2)$$

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall\, x \in \Sigma^*$

- base case: $|x| = 0$ and thus $x = \epsilon$

$$\widehat{\delta_3}((p,q),x) = (p,q) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x))$$

- induction step: $|x| > 0$ and thus $x = ya$ with $|y| = |x| - 1$

$$\begin{aligned}
\widehat{\delta_3}((p,q),x) &= \delta_3(\widehat{\delta_3}((p,q),y),a) && \text{(definition of } \widehat{\delta_3}\text{)} \\
&= \delta_3((\widehat{\delta_1}(p,y), \widehat{\delta_2}(q,y)),a) && \text{(induction hypothesis)} \\
&= (\delta_1(\widehat{\delta_1}(p,y),a), \delta_2(\widehat{\delta_2}(q,y),a)) && \text{(definition of } \delta_3\text{)} \\
&= (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) && \text{(definition of } \widehat{\delta_1} \text{ and } \widehat{\delta_2}\text{)}
\end{aligned}$$

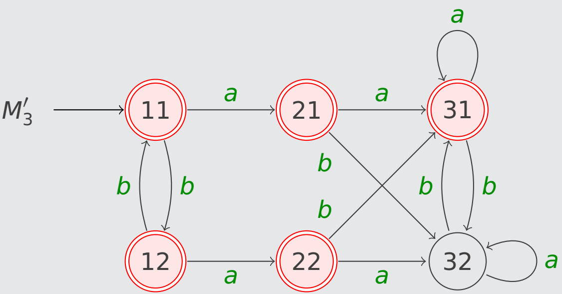
Theorem

regular sets are effectively closed under complement

Proof

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
- $\sim A = \Sigma^* - A = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ with
  1. $Q_2 = Q_1$
  2. $\delta_2(q, a) = \delta_1(q, a) \quad \forall\, q \in Q_2\ \forall\, a \in \Sigma$
  3. $s_2 = s_1$
  4. $F_2 = Q_1 - F_1$

Example



$$L(M_3') = \sim L(M_3)$$

Theorem

regular sets are effectively closed under union

Proof (explicit construction)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

- $A \cup B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with

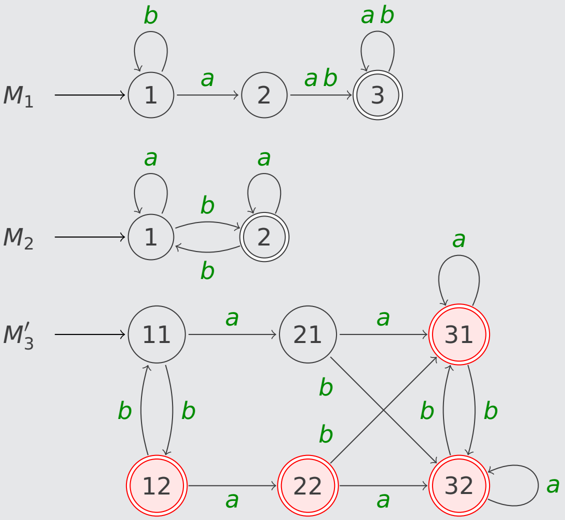
  ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$

  ② $F_3 = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

  ③ $s_3 = (s_1, s_2)$

  ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a)) \quad \forall\, p \in Q_1\ \forall\, q \in Q_2\ \forall\, a \in \Sigma$

Example



$$L(M_3') = L(M_1) \cup L(M_2)$$

Outline

Kozen

- Lectures 1 – 4

Important Concepts

- alphabet
- closure properties
- DFA
- language
- product construction
- regular set
- string

homework for October 11

Solutions

... must be uploaded (PDF format) in OLAT before 7 am on Friday

... bonus exercises give bonus points