universität innsbruck

WS 2024 lecture 2

Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with

  ① Q: finite set of states

  ② Σ: input alphabet

  ③ $\delta \colon Q \times \Sigma \to Q$: transition function

  ④ $s \in Q$: start state

  ⑤ $F \subseteq Q$: final (accept) states

- $\widehat{\delta} \colon Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \epsilon) = q \qquad \widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is accepted by M if $\widehat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M: $L(M) = \{x \mid \widehat{\delta}(s, x) \in F\}$

Definition

set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Theorem

regular sets are effectively closed under intersection, union, and complement

Automata

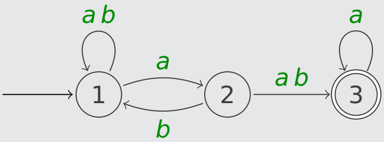
- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Example



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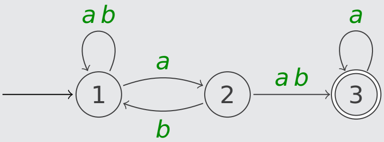
Example

NFA $M = (Q, \Sigma, \Delta, S, F)$



1. $Q = \{1, 2, 3\}$
2. $\Sigma = \{a, b\}$
3. $\Delta : Q \times \Sigma \to 2^Q$
4. $S = \{1\}$
5. $F = \{3\}$

Δ	a	b
1	$\{1, 2\}$	$\{1\}$
2	$\{3\}$	$\{1, 3\}$
3	$\{3\}$	$\varnothing$

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- $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

Theorem

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Proof

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- claim: $\widehat{\delta_M}(A, x) = \widehat{\Delta}_N(A, x)$ for all $A \subseteq Q_N$ and $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$

Theorem

every set accepted by NFA is regular

Proof (subset construction)

- NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- $L(N) = L(M)$ for DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ with

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  ② $\delta_M(A, a) = \widehat{\Delta}_N(A, a)$ for all $A \subseteq Q_N$ and $a \in \Sigma$

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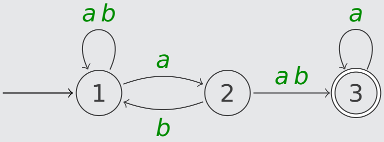
Example



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$B = \{1\} \quad F = \{1,3\}$

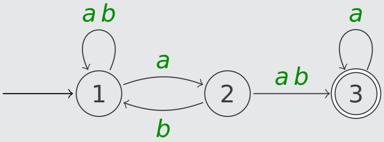
$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

	a	b
A		
B		
C		
D		

	a	b
E		
F		
G		
H		

Example



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	a	*b*
A	*A*	
B		
C		
D		

	a	*b*
E		
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G		
H		

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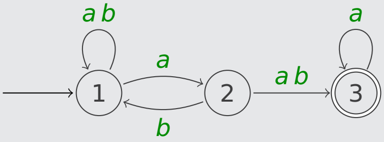

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A	A	A
B		
C		
D		

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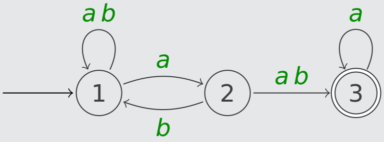

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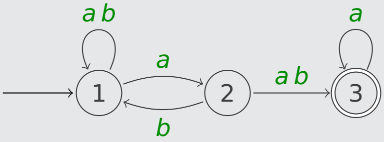

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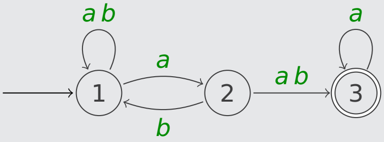

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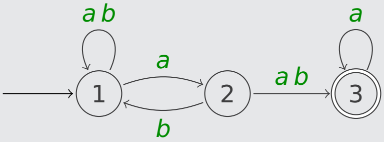

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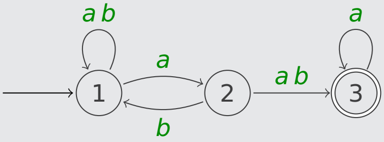

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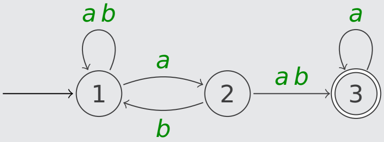

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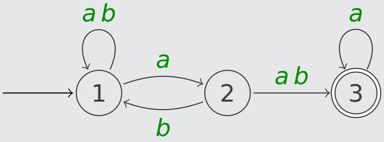

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	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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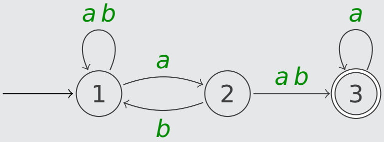

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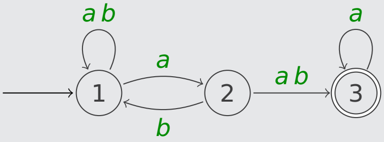

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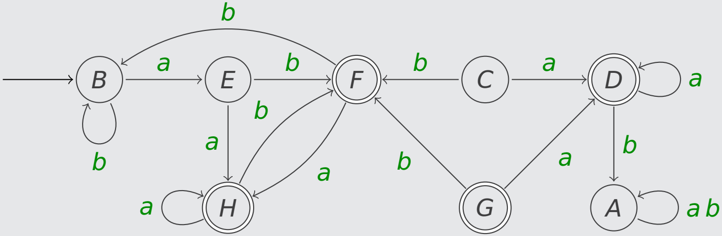

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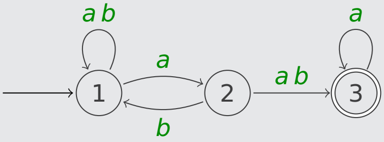

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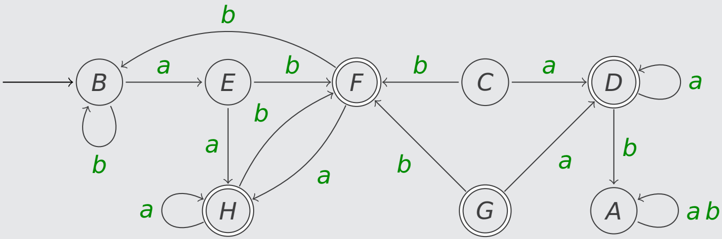

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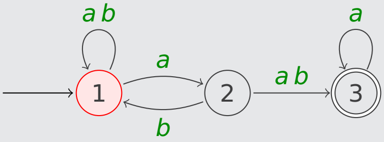

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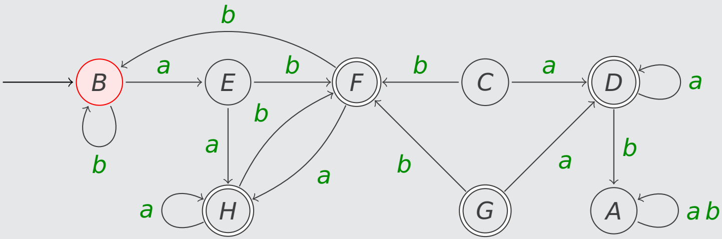

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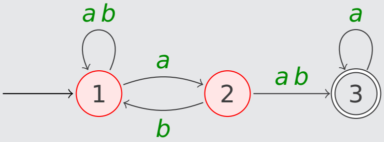

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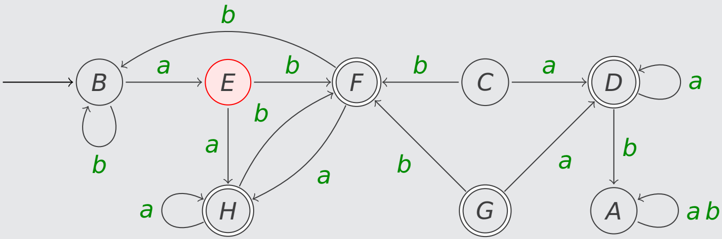

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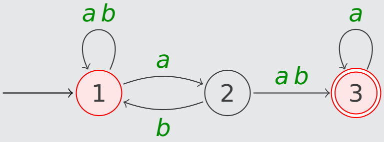

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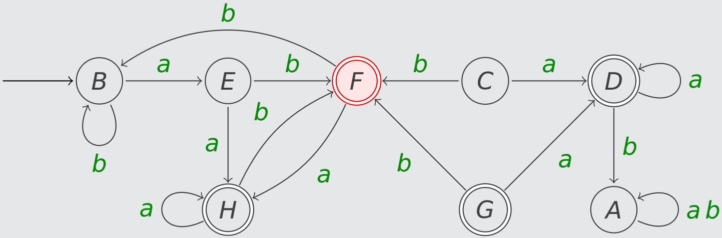

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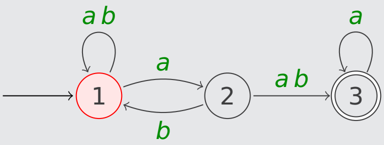

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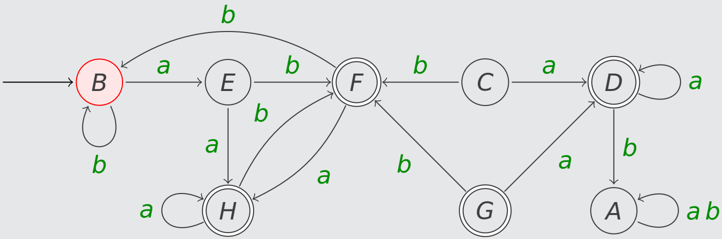

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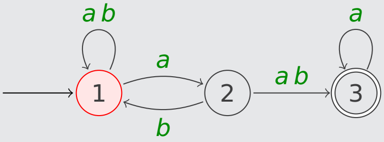

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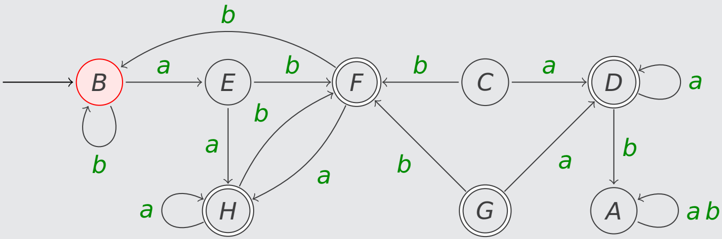

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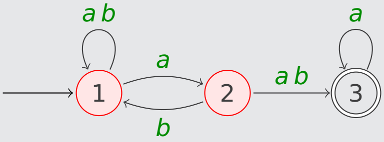

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	a	b
E	H	F
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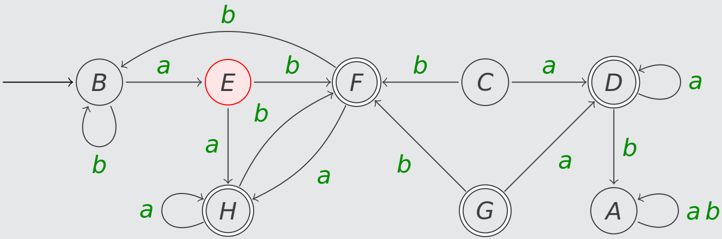
H	H	F

abbbaababbabbbaababba



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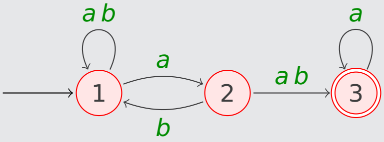


$A = \varnothing \quad E = \{1,2\}$

$B = \{1\} \quad F = \{1,3\}$

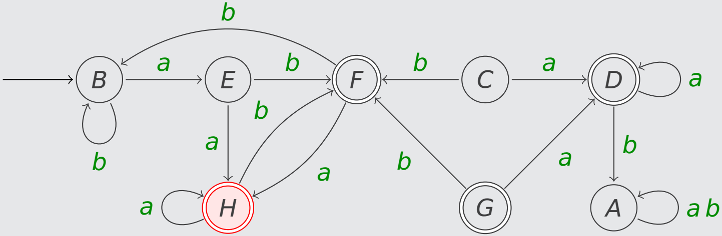
$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

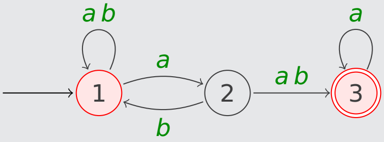
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing$ $\quad E = \{1, 2\}$

$B = \{1\}$ $\quad F = \{1, 3\}$

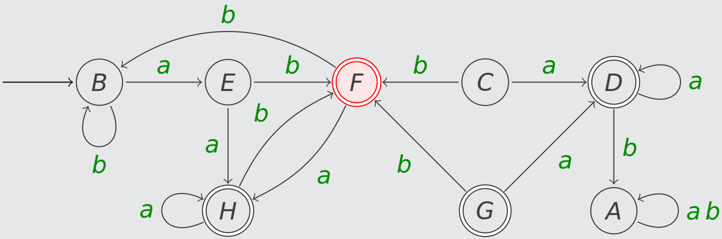
$C = \{2\}$ $\quad G = \{2, 3\}$

$D = \{3\}$ $\quad H = \{1, 2, 3\}$

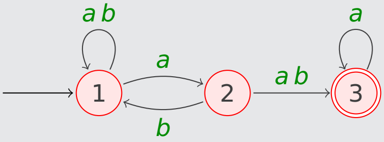
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing$ $E = \{1,2\}$

$B = \{1\}$ $F = \{1,3\}$

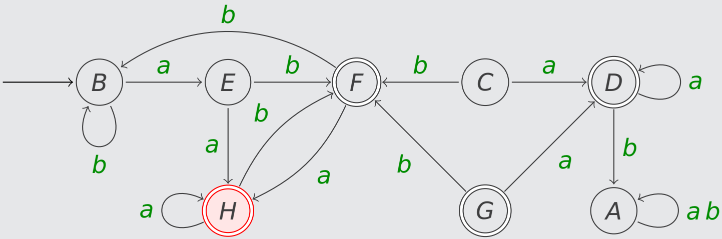
$C = \{2\}$ $G = \{2,3\}$

$D = \{3\}$ $H = \{1,2,3\}$

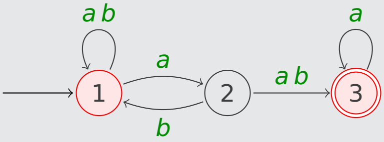
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing \quad E = \{1,2\}$

$B = \{1\} \quad F = \{1,3\}$

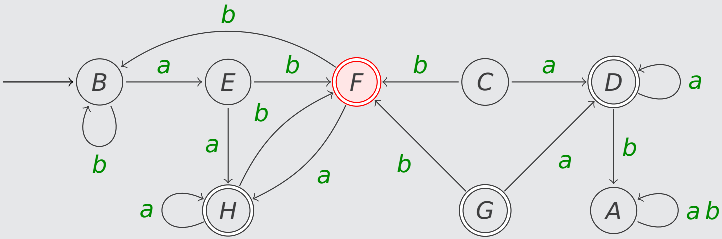
$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

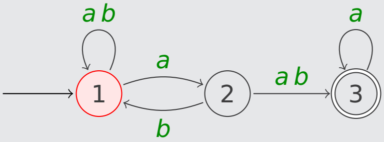
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing \quad E = \{1,2\}$

$B = \{1\} \quad F = \{1,3\}$

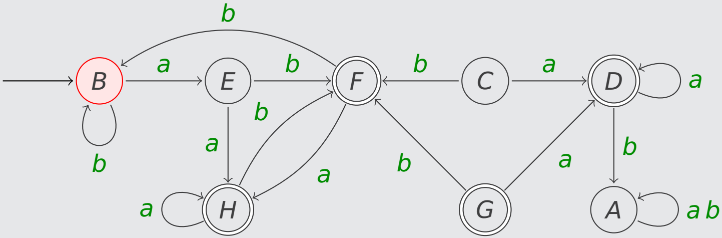
$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

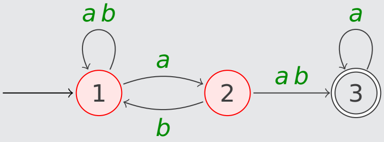
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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$B = \{1\} \quad F = \{1,3\}$

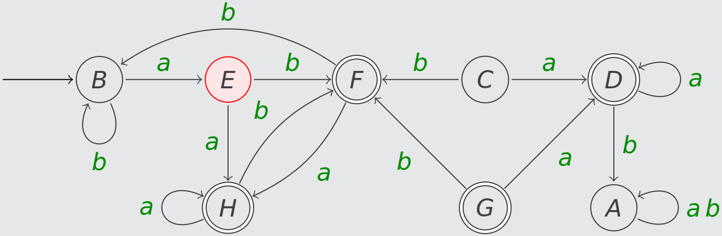
$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

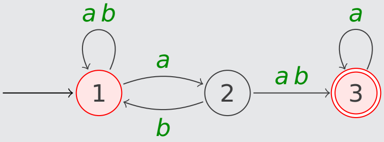
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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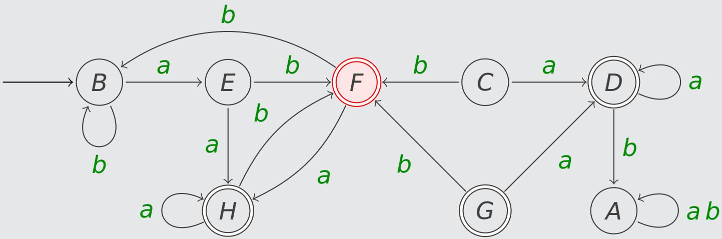
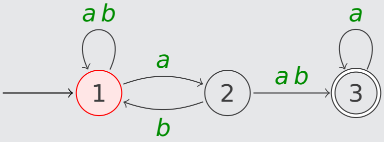
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing \quad E = \{1,2\}$

$B = \{1\} \quad F = \{1,3\}$

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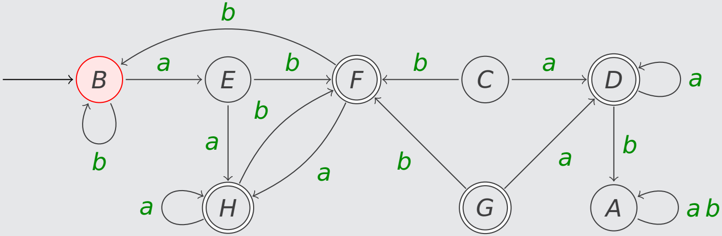
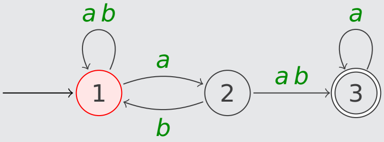
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing$ $\quad E = \{1,2\}$

$B = \{1\}$ $\quad F = \{1,3\}$

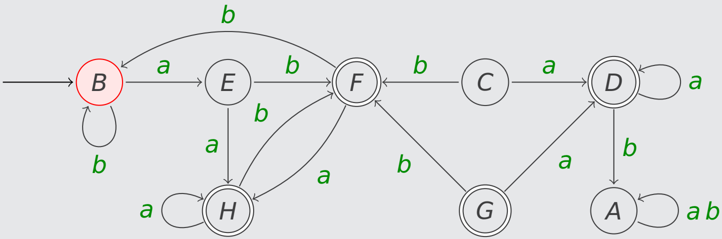
$C = \{2\}$ $\quad G = \{2,3\}$

$D = \{3\}$ $\quad H = \{1,2,3\}$

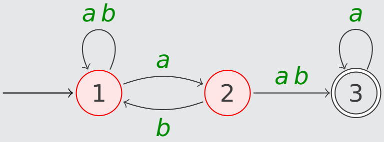
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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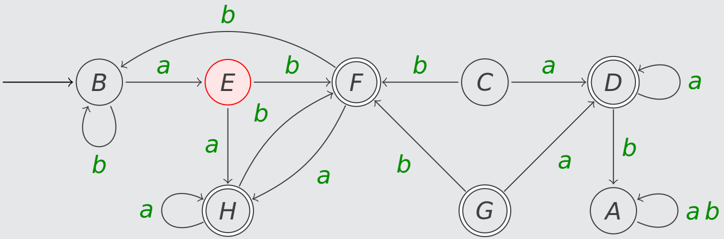
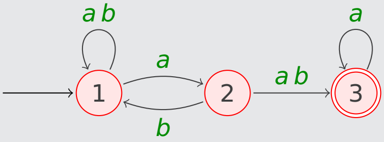


Example



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$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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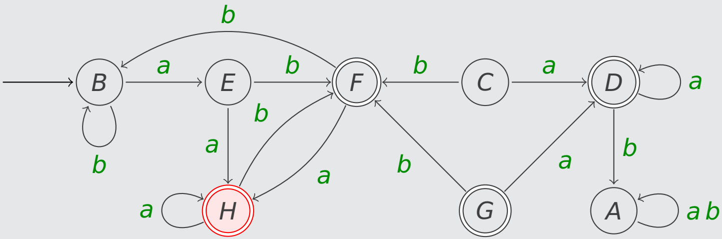
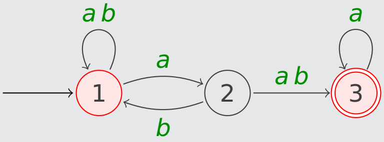


Example



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$C = \{2\}$ $G = \{2, 3\}$

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	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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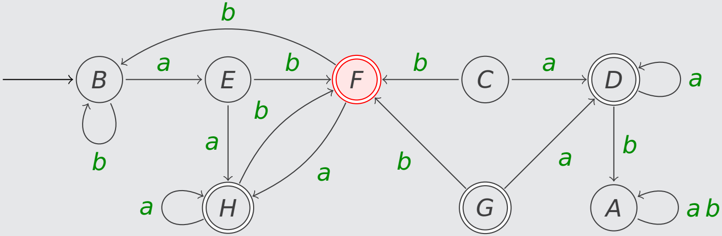
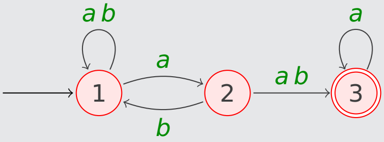


Example



$A = \varnothing$ $\quad E = \{1,2\}$

$B = \{1\}$ $\quad F = \{1,3\}$

$C = \{2\}$ $\quad G = \{2,3\}$

$D = \{3\}$ $\quad H = \{1,2,3\}$

	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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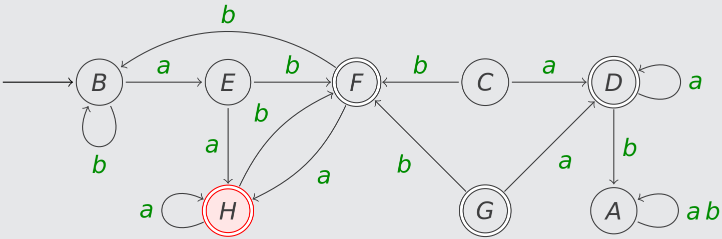
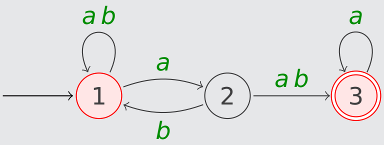


Example



$A = \varnothing \quad E = \{1,2\}$

$B = \{1\} \quad F = \{1,3\}$

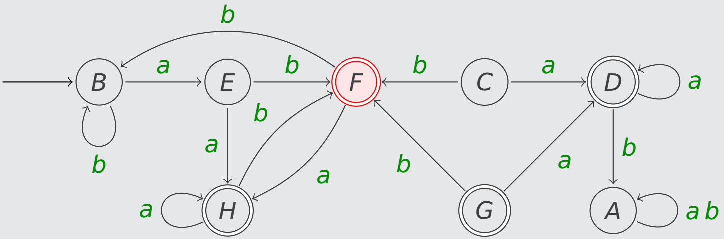
$C = \{2\} \quad G = \{2,3\}$

$D = \{3\} \quad H = \{1,2,3\}$

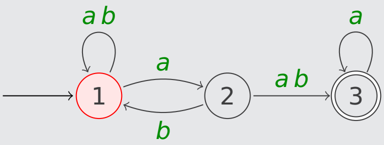
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing$ $E = \{1,2\}$

$B = \{1\}$ $F = \{1,3\}$

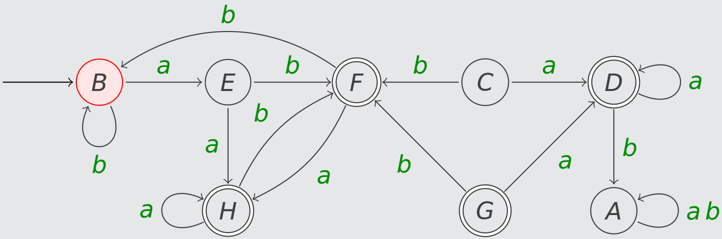
$C = \{2\}$ $G = \{2,3\}$

$D = \{3\}$ $H = \{1,2,3\}$

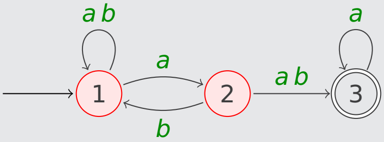
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example

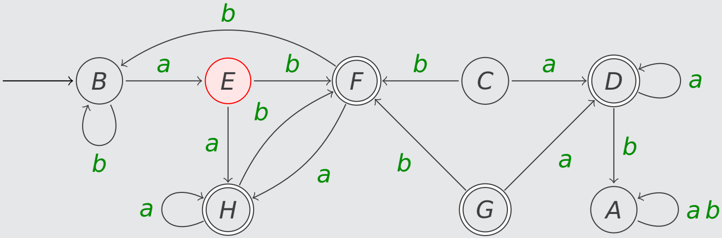


$A = \varnothing$	$E = \{1,2\}$
$B = \{1\}$	$F = \{1,3\}$
$C = \{2\}$	$G = \{2,3\}$
$D = \{3\}$	$H = \{1,2,3\}$

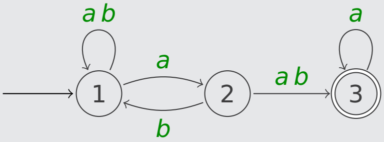
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing \quad E = \{1,2\}$

$B = \{1\} \quad F = \{1,3\}$

$C = \{2\} \quad G = \{2,3\}$

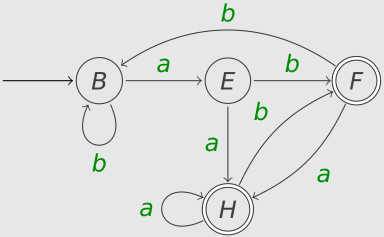
$D = \{3\} \quad H = \{1,2,3\}$

	a	b
A	A	A
B	E	B
C	D	F
D	D	A

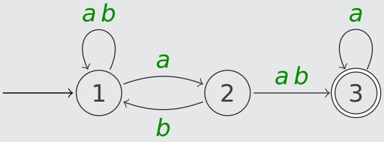
	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba

remove inaccesible states



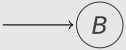
Example



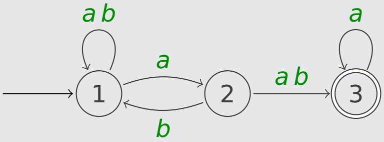
$B = \{1\}$

	a	b

	a	b



Example

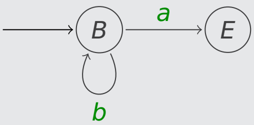


$B = \{1\}$

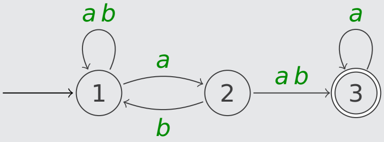
$E = \{1, 2\}$

	a	b
B	E	B

	a	b



Example



$B = \{1\}$

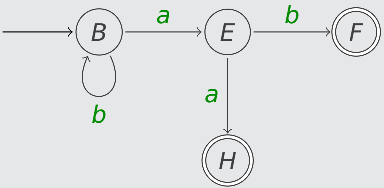
$E = \{1, 2\}$

$F = \{1, 3\}$

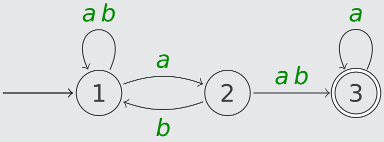
$H = \{1, 2, 3\}$

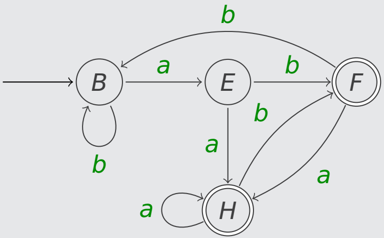
	a	b
B	E	B

	a	b
E	H	F



Example





$B = \{1\}$

$E = \{1,2\}$

$F = \{1,3\}$

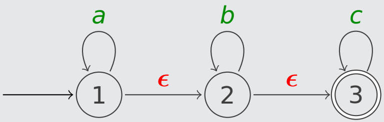
$H = \{1,2,3\}$

	a	b
B	E	B

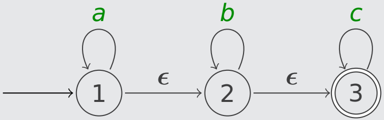
	a	b
E	H	F
F	H	B
H	H	F

Outline

Example



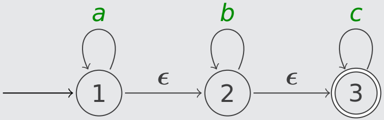
Example



Definitions

- NFA with ϵ-transitions (NFA$_\epsilon$) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that

Example

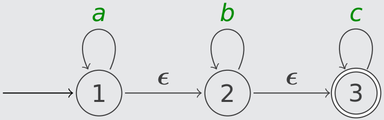


Definitions

- NFA with ϵ-transitions (NFA$_\epsilon$) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that

  ① $\epsilon \notin \Sigma$

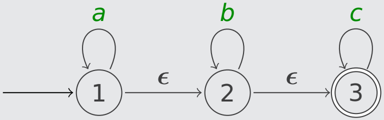
Example



Definitions

- NFA with ϵ-transitions (NFA_ϵ) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that
  ① $\epsilon \notin \Sigma$
  ② $N_\epsilon = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$

Example



Definitions

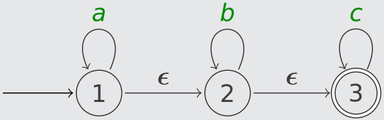
- NFA with ϵ-transitions (NFA_ϵ) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that

  ① $\epsilon \notin \Sigma$

  ② $N_\epsilon = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$

- ϵ-closure of set $A \subseteq Q$ is defined as $C_\epsilon(A) = \bigcup \{ \widehat{\Delta}_{N_\epsilon}(A, x) \mid x \in \{\epsilon\}^* \}$

Example



$C_\epsilon(\{1\}) = \{1,2,3\}$

Definitions

- NFA with ϵ-transitions (NFA_ϵ) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that
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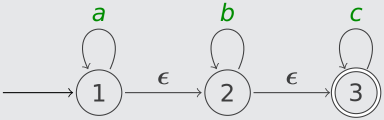

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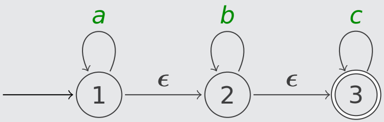

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$C_\epsilon(A)$ is least extension of A that is closed under ϵ-transitions:

$$q \in C_\epsilon(A) \quad \Longrightarrow \quad \Delta_{N_\epsilon}(q, \epsilon) \subseteq C_\epsilon(A)$$

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Proof (construction)

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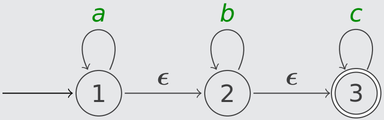
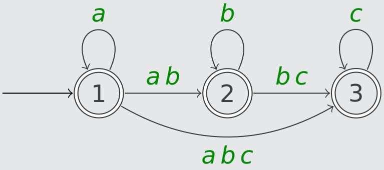
Example

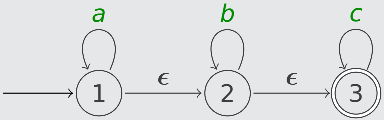


Example



Example

▶

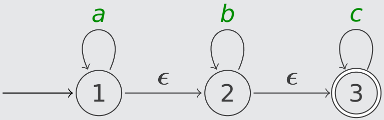


Example

NFA$_\epsilon$ $N_1 = (\{1,2,3\},\{a,b,c\},\epsilon,\Delta_1,\{1\},\{3\})$ with

-

Δ_1	a	b	c	ϵ
1	$\{1\}$	$\varnothing$	$\varnothing$	$\{2\}$
2	$\varnothing$	$\{2\}$	$\varnothing$	$\{3\}$
3	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$



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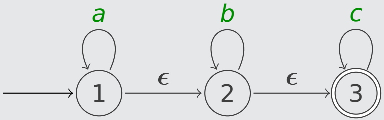

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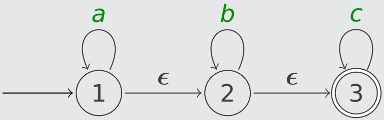

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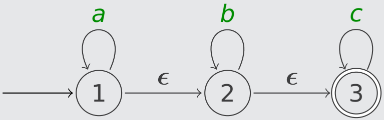

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-

Δ_2	a	b	c
1			
2			
3			

Example

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Δ_1	a	b	c	ϵ
1	$\{1\}$	$\varnothing$	$\varnothing$	$\{2\}$
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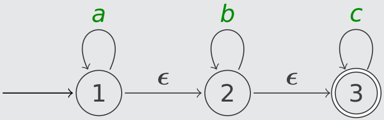
3	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$



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Δ_2	a	b	c
1			
2			
3			

$$\Delta_2(1,a) = \widehat{\Delta}_1(\{1\},a)$$

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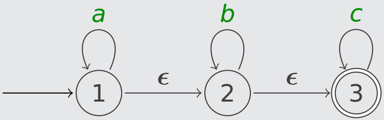

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2			
3			

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NFA$_\epsilon$ $N_1 = (\{1,2,3\},\{a,b,c\},\epsilon,\Delta_1,\{1\},\{3\})$ with

- Δ_1

Δ_1	a	b	c	ϵ
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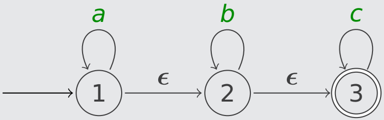

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- Δ_2

Δ_2	a	b	c
1			
2			
3			

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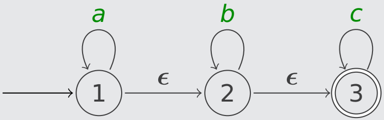

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2			
3			

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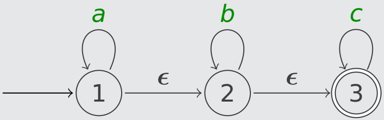

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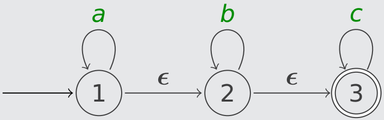

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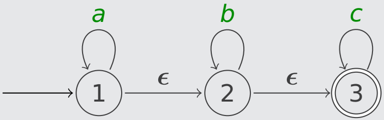

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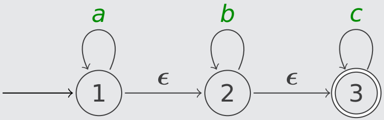

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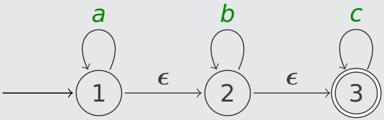

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$$= C_\epsilon(\quad \varnothing \quad) \cup C_\epsilon(\quad \{2\} \quad) \cup C_\epsilon(\quad \varnothing \quad)$$

Example

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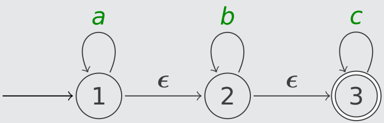

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2			
3			

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Example

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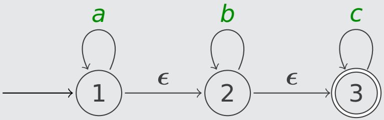
3	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$



NFA $N_2 = (\{1,2,3\}, \{a,b,c\}, \Delta_2, \{1\}, F_2)$ with

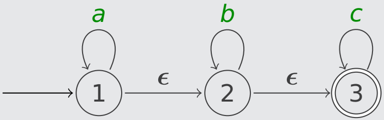
- $F_2 = \{q \mid C_\epsilon(\{q\}) \cap \{3\} \neq \varnothing\} = \{q \mid 3 \in C_\epsilon(\{q\})\} = \{1,2,3\}$
Δ_2	a	b	c
1	$\{1,2,3\}$	$\{2,3\}$	
2			
3			

$$\Delta_2(3,b) = \widehat{\Delta}_1(\{3\},b) = \bigcup \{C_\epsilon(\Delta_1(q,b)) \mid q \in \widehat{\Delta}_1(\{3\},\epsilon)\}$$
$$= C_\epsilon(\Delta_1(3,b))$$

Example

NFA$_\epsilon$ $N_1 = (\{1,2,3\},\{a,b,c\},\epsilon,\Delta_1,\{1\},\{3\})$ with

Δ_1	a	b	c	ϵ
1	$\{1\}$	$\varnothing$	$\varnothing$	$\{2\}$
2	$\varnothing$	$\{2\}$	$\varnothing$	$\{3\}$
3	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$



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1	$\{1\}$	$\varnothing$	$\varnothing$	$\{2\}$
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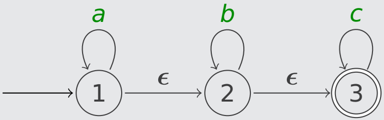

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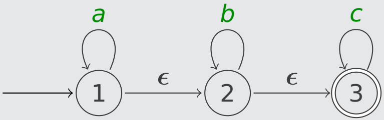

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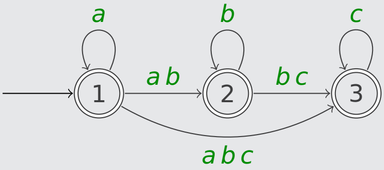

Outline

Question

What is the language accepted by the NFA_ϵ given by the following transition table ?

		ϵ	a	b
$\rightarrow$	1	$\{2\}$	$\{1,2\}$	$\{1\}$
	2	$\varnothing$	$\varnothing$	$\{3\}$
	3	$\varnothing$	$\{4\}$	$\{4\}$
	4 F	$\varnothing$	$\varnothing$	$\varnothing$

A $\{a,b\}^*$

B $\{xaby \mid x \in \{a,b\}^* \text{ and } y \in \{a,b\}\}$

C $\{xyz \mid x,z \in \{a,b\}^* \text{ and } y \in \{a,b\}\}$

D $\{xyz \mid x \in \{a,b\}^*,\ y \in \{b,ab\} \text{ and } z \in \{a,b\}\}$

Outline

Theorem

regular sets are effectively closed under union, concatenation, and asterate

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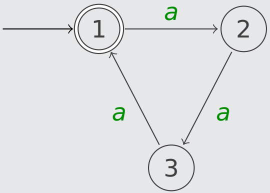
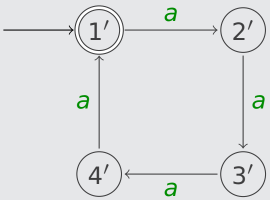
Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$

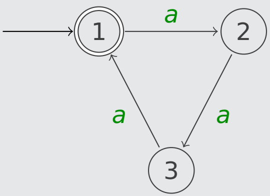


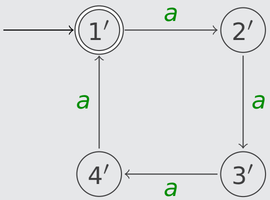
$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



Example

$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\} \quad \cup \quad \{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$$





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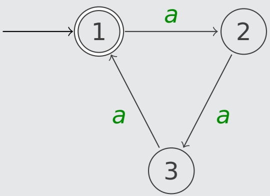

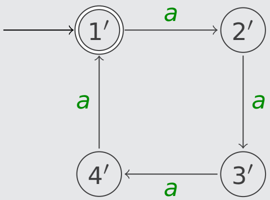

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3 \text{ or } 4\}$

Theorem

regular sets are effectively closed under union, concatenation, and asterate

Proof

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

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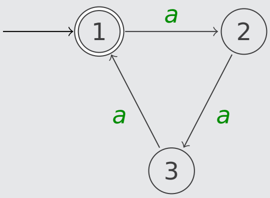
$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$



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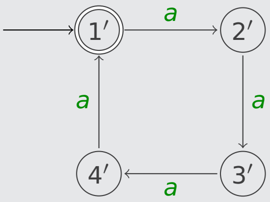

Example

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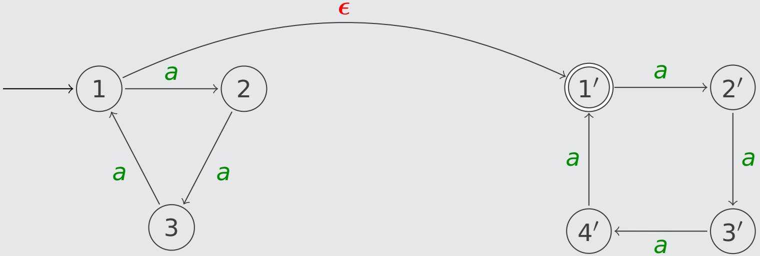

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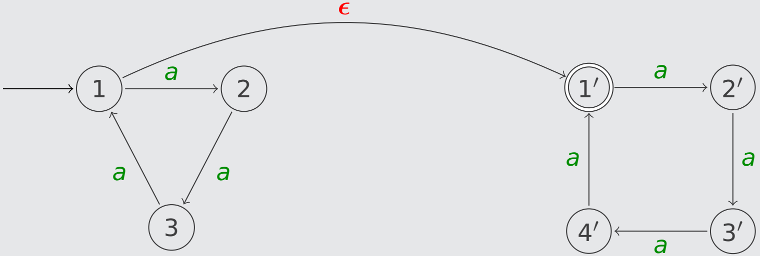

$\{x \in \{a\}^* \mid |x| \notin \{1, 2, 5\}\}$

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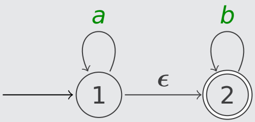
Example

$$\{a\}^*\{b\}^*$$



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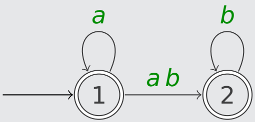
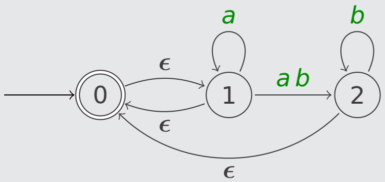


Example

$$(\{a\}^*\{b\}^*)^* = \{a,b\}^*$$



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Outline

Definitions

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$\Delta_N((p,2),a) = \{(q,2) \mid \delta_M(p,a) = q\}$ for all $a \in \Sigma$

③ $S_N = \{(s_M, 0)\}$

④ $F_N = F_M \times \{0,1,2\}$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad \Longleftrightarrow \quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

for all $p, q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \iff \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

for all $p,q \in Q_M$, $y \in \{0,1\}^*$, $i,j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

- $N_2(A) = \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\}$

Proof (cont'd)

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  $= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\}$

Proof (cont'd)

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  $= \{y \mid H(y,x) = k \text{ and } \widehat{\delta_M}(s_M,x) = q \text{ for some } x \in A,\ k \in \{0,1,2\} \text{ and } q \in F_M\}$

Proof (cont'd)

- key property:

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$$= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } q \in F_M \text{ and } k \in \{0,1,2\}\}$$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \iff \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

for all $p, q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

- $$\begin{aligned} N_2(A) &= \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\} \\ &= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid H(y,x) = k \text{ and } \widehat{\delta_M}(s_M,x) = q \text{ for some } x \in A,\ k \in \{0,1,2\} \text{ and } q \in F_M\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } q \in F_M \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } (q,k) \in F_N\} \end{aligned}$$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad \Longleftrightarrow \quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

  for all $p, q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

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Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \iff \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

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Outline

Kozen

- Lecture 5 and 6

Kozen

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Important Concepts

- ϵ-transition
- ϵ-closure
- asterate
- Hamming distance
- NFA
- NFA_ϵ
- subset construction

Kozen

- Lecture 5 and 6

Important Concepts

- ϵ-transition
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homework for October 25