universität innsbruck

WS 2024 lecture 2

Automata and Logic

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Outline

Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\delta \colon Q \times \Sigma \to Q$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\delta} \colon Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \epsilon) = q \qquad\qquad \widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is accepted by M if $\widehat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M: $L(M) = \{x \mid \widehat{\delta}(s, x) \in F\}$

Definition

set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Theorem

regular sets are effectively closed under intersection, union, and complement

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

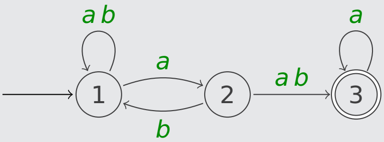
Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Example

NFA $M = (Q, \Sigma, \Delta, S, F)$



1. $Q = \{1, 2, 3\}$
2. $\Sigma = \{a, b\}$
3. $\Delta : Q \times \Sigma \to 2^Q$
4. $S = \{1\}$
5. $F = \{3\}$

Δ	a	b
1	$\{1, 2\}$	$\{1\}$
2	$\{3\}$	$\{1, 3\}$
3	$\{3\}$	$\varnothing$

Definitions

- nondeterministic finite automaton (NFA) is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with

  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\Delta\colon Q \times \Sigma \to 2^Q$: transition function
  4. $S \subseteq Q$: set of start states
  5. $F \subseteq Q$: final (accept) states

- $\widehat{\Delta}\colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}(A, \epsilon) = A \qquad\qquad \widehat{\Delta}(A, xa) = \bigcup_{q \in \widehat{\Delta}(A,x)} \Delta(q, a)$$

- $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

Theorem

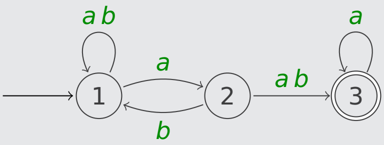
every set accepted by NFA is regular

Proof (subset construction)

- NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- $L(N) = L(M)$ for DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ with
  ① $Q_M = 2^{Q_N}$
  ② $\delta_M(A, a) = \widehat{\Delta}_N(A, a)$ for all $A \subseteq Q_N$ and $a \in \Sigma$
  ③ $s_M = S_N$
  ④ $F_M = \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\}$
- claim: $\widehat{\delta_M}(A, x) = \widehat{\Delta}_N(A, x)$ for all $A \subseteq Q_N$ and $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$

Example



$A = \varnothing$ $\quad E = \{1,2\}$

$B = \{1\}$ $\quad \color{red}{F} = \{1,3\}$

$C = \{2\}$ $\quad \color{red}{G} = \{2,3\}$

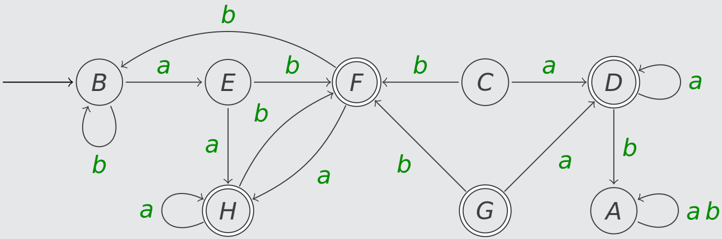
$\color{red}{D} = \{3\}$ $\quad \color{red}{H} = \{1,2,3\}$

	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

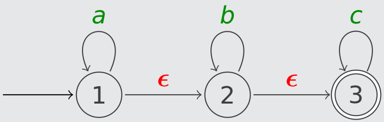
abbbaababbabbbaababba

remove inaccesible states



Outline

Example



$C_\epsilon(\{1\}) = \{1,2,3\}$

$\widehat{\Delta}(\{1\}, b) = \{2,3\}$

Definitions

- NFA with ϵ-transitions (NFA$_\epsilon$) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that
  1. $\epsilon \notin \Sigma$
  2. $N_\epsilon = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$
- ϵ-closure of set $A \subseteq Q$ is defined as $C_\epsilon(A) = \bigcup \{ \widehat{\Delta}_{N_\epsilon}(A, x) \mid x \in \{\epsilon\}^* \}$
- $\widehat{\Delta}_N \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}_N(A, \epsilon) = C_\epsilon(A) \qquad \widehat{\Delta}_N(A, xa) = \bigcup \{ C_\epsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, x) \}$$

Lemma

$C_\epsilon(A)$ is least extension of A that is closed under ϵ-transitions:

$$q \in C_\epsilon(A) \quad \Longrightarrow \quad \Delta_{N_\epsilon}(q, \epsilon) \subseteq C_\epsilon(A)$$

Theorem

every set accepted by NFA_ϵ is regular

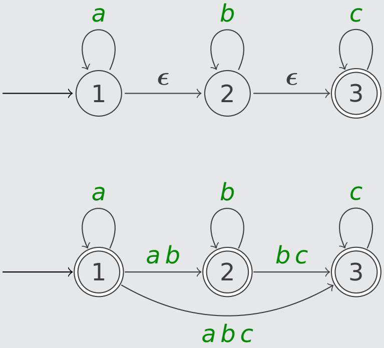
Proof (construction)

- NFA_ϵ $N_1 = (Q, \Sigma, \epsilon, \Delta_1, S, F_1)$
- $L(N_1) = L(N_2)$ for NFA $N_2 = (Q, \Sigma, \Delta_2, S, F_2)$ with

  ① $\Delta_2(q, a) = \widehat{\Delta}_1(\{q\}, a)$ for all $q \in Q$ and $a \in \Sigma$

  ② $F_2 = \{q \mid C_\epsilon(\{q\}) \cap F_1 \neq \varnothing\}$

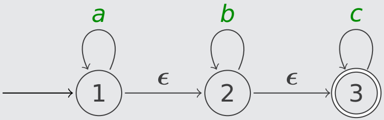
Example



Example

NFA$_\epsilon$ $N_1 = (\{1,2,3\},\{a,b,c\},\epsilon,\Delta_1,\{1\},\{3\})$ with

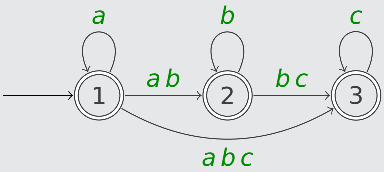
Δ_1	a	b	c	ϵ
1	$\{1\}$	$\varnothing$	$\varnothing$	$\{2\}$
2	$\varnothing$	$\{2\}$	$\varnothing$	$\{3\}$
3	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$



NFA $N_2 = (\{1,2,3\},\{a,b,c\},\Delta_2,\{1\},F_2)$ with

- $F_2 = \{q \mid C_\epsilon(\{q\}) \cap \{3\} \neq \varnothing\} = \{q \mid 3 \in C_\epsilon(\{q\})\} = \{1,2,3\}$

Δ_2	a	b	c
1	$\{1,2,3\}$	$\{2,3\}$	$\{3\}$
2	$\varnothing$	$\{2,3\}$	$\{3\}$
3	$\varnothing$	$\varnothing$	$\{3\}$



Outline

Question

What is the language accepted by the NFA_ϵ given by the following transition table ?

		ϵ	a	b
$\rightarrow$	1	$\{2\}$	$\{1,2\}$	$\{1\}$
	2	$\varnothing$	$\varnothing$	$\{3\}$
	3	$\varnothing$	$\{4\}$	$\{4\}$
	$4\,F$	$\varnothing$	$\varnothing$	$\varnothing$

A $\{a,b\}^*$

B $\{xaby \mid x \in \{a,b\}^* \text{ and } y \in \{a,b\}\}$

C $\{xyz \mid x,z \in \{a,b\}^* \text{ and } y \in \{a,b\}\}$

D $\{xyz \mid x \in \{a,b\}^*,\ y \in \{b,ab\} \text{ and } z \in \{a,b\}\}$

Outline

Theorem

regular sets are effectively closed under union, concatenation, and asterate

Proof

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

  $B = L(N_2)$ for NFA $N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- without loss of generality $Q_1 \cap Q_2 = \varnothing$
- $A \cup B = L(N)$ for NFA $N = (Q, \Sigma, \Delta, S, F)$ with

  ① $Q = Q_1 \cup Q_2$

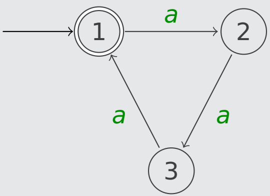
  ② $S = S_1 \cup S_2$

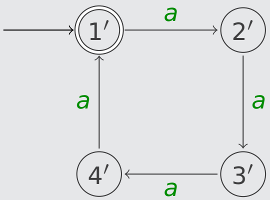
  ③ $F = F_1 \cup F_2$

  ④ $\Delta(q,a) = \begin{cases} \Delta_1(q,a) & \text{if } q \in Q_1 \\ \Delta_2(q,a) & \text{if } q \in Q_2 \end{cases}$

Example

$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\} \quad \cup \quad \{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$$





$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3 \text{ or } 4\}$$

Theorem

regular sets are effectively closed under union, concatenation, and asterate

Proof

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

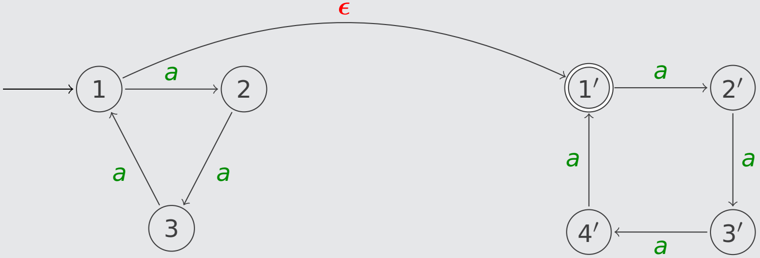
  $B = L(N_2)$ for NFA $N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- without loss of generality $Q_1 \cap Q_2 = \varnothing$
- $AB = L(N)$ for NFA$_\epsilon$ $N = (Q, \Sigma, \epsilon, \Delta, S_1, F_2)$ with

  ① $Q = Q_1 \cup Q_2$

  ② $$\Delta(q,a) = \begin{cases} \Delta_1(q,a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ \Delta_2(q,a) & \text{if } q \in Q_2 \text{ and } a \in \Sigma \\ S_2 & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \varnothing & \text{otherwise} \end{cases}$$

Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$ $\quad$ $\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



$\{x \in \{a\}^* \mid |x| \notin \{1, 2, 5\}\}$

Theorem

regular sets are effectively closed under union, concatenation, and asterate

Proof

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
- $A^* = L(N)$ for NFA$_\epsilon$ $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ with

① $Q = Q_1 \uplus \{s\}$

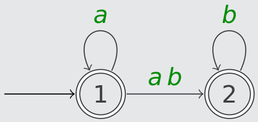
② $S = \{s\}$

③ $F = \{s\}$

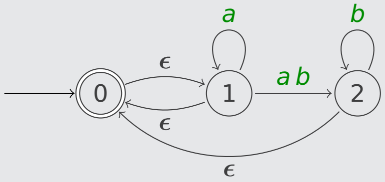
④ $$\Delta(q,a) = \begin{cases} \Delta_1(q,a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ S_1 & \text{if } q = s \text{ and } a = \epsilon \\ S & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \varnothing & \text{otherwise} \end{cases}$$

Example

$\{a\}^*\{b\}^*$



$(\{a\}^*\{b\}^*)^* = \{a,b\}^*$



Outline

Definitions

- Hamming distance $H(x, y)$ is number of places where bit strings x and y differ
- if $|x| \neq |y|$ then $H(x, y) = \infty$
- $N_k(A) = \{x \in \{0,1\}^* \mid H(x, y) \leqslant k$ for some $y \in A\}$

Lemma

$A \subseteq \{0,1\}^*$ is regular $\implies$ $N_2(A)$ is regular

Lemma

$A \subseteq \{0,1\}^*$ is regular $\implies$ $N_2(A)$ is regular

Proof

- $A = L(M)$ for DFA $M = (Q_M, \{0,1\}, \delta_M, s_M, F_M)$
- define NFA $N = (Q_N, \{0,1\}, \Delta_N, S_N, F_N)$ with

① $Q_N = Q_M \times \{0,1,2\}$

② $\Delta_N((p,0),a) = \{(q,0) \mid \delta_M(p,a) = q\} \cup \{(q,1) \mid \delta_M(p,b) = q \text{ for some } b \neq a\}$

$\Delta_N((p,1),a) = \{(q,1) \mid \delta_M(p,a) = q\} \cup \{(q,2) \mid \delta_M(p,b) = q \text{ for some } b \neq a\}$

$\Delta_N((p,2),a) = \{(q,2) \mid \delta_M(p,a) = q\}$ for all $a \in \Sigma$

③ $S_N = \{(s_M, 0)\}$

④ $F_N = F_M \times \{0,1,2\}$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \iff \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

for all $p,q \in Q_M$, $y \in \{0,1\}^*$, $i,j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

- $N_2(A) = \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\}$

$$= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\}$$

$$= \{y \mid H(y,x) = k \text{ and } \widehat{\delta_M}(s_M,x) = q \text{ for some } x \in A,\ k \in \{0,1,2\} \text{ and } q \in F_M\}$$

$$= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } q \in F_M \text{ and } k \in \{0,1,2\}\}$$

$$= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } (q,k) \in F_N\}$$

$$= \{y \mid \widehat{\Delta}_N(\{(s_M,0)\},y) \cap F_N \neq \varnothing\}$$

$$= L(N)$$

Outline

Kozen

- Lecture 5 and 6

Important Concepts

- ϵ-transition
- ϵ-closure
- asterate
- Hamming distance
- NFA
- NFA_{ϵ}
- subset construction

homework for October 25