universität innsbruck

WS 2024 lecture 3

Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

1. **Summary of Previous Lecture**

2. **Regular Expressions**

3. **Intermezzo**

4. **Homomorphisms**

5. **Decision Problems**

6. **Further Reading**

Definitions

- nondeterministic finite automaton (NFA) is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with

  ① Q: finite set of states

  ② Σ: input alphabet

  ③ $\Delta : Q \times \Sigma \to 2^Q$: transition function

  ④ $S \subseteq Q$: set of start states

  ⑤ $F \subseteq Q$: final (accept) states

- $\widehat{\Delta} : 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}(A, \epsilon) = A \qquad \widehat{\Delta}(A, xa) = \bigcup_{q \in \widehat{\Delta}(A,x)} \Delta(q, a)$$

- $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

Theorem

every set accepted by NFA is regular

Definitions

- NFA with ϵ-transitions (NFA$_\epsilon$) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that
  1. $\epsilon \notin \Sigma$
  2. $M_\epsilon = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$
- ϵ-closure of set $A \subseteq Q$ is defined as $C_\epsilon(A) = \bigcup \{ \widehat{\Delta}_{N_\epsilon}(A, x) \mid x \in \{\epsilon\}^* \}$
- $\widehat{\Delta}_N \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}_N(A, \epsilon) = C_\epsilon(A) \qquad\qquad \widehat{\Delta}_N(A, xa) = \bigcup \{ C_\epsilon(\Delta(q, a)) \mid q \in \widehat{\Delta}_N(A, x) \}$$

Theorem

- every set accepted by NFA$_\epsilon$ is regular
- regular sets are effectively closed under union, concatenation, and asterate

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

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$$a \in \Sigma \qquad \epsilon \qquad \varnothing \qquad \beta + \gamma \qquad \beta\gamma \qquad \beta^*$$

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Example

regular expression $(a+b)^*b$ matches all strings over $\Sigma = \{a,b\}$ that end with b

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Definition

regular expressions α and β are equivalent $(\alpha \equiv \beta)$ if $L(\alpha) = L(\beta)$

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finite automata and regular expressions are equivalent

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induction on regular expression α

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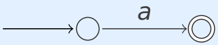
$a \in \Sigma$	$\{a\}$	
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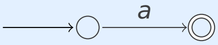
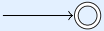
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
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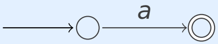
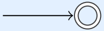
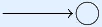
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
$\varnothing$	$\varnothing$	

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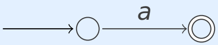
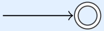
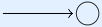
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
$\varnothing$	$\varnothing$	

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$L(\beta)$ and $L(\gamma)$ are regular according to induction hypothesis

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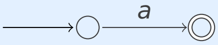
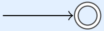
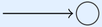
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
$\varnothing$	$\varnothing$	

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$L(\beta)$ and $L(\gamma)$ are regular according to induction hypothesis

$\Longrightarrow$ $L(\alpha)$ is regular according to closure properties of regular sets

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    - $k = 0 \implies x = y$

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Theorem

finite automata and regular expressions are equivalent:

$$\text{for all } A \subseteq \Sigma^* \quad A \text{ is regular} \quad \Longleftrightarrow \quad A = L(\alpha) \text{ for some regular expression } \alpha$$

Proof ($\Longrightarrow$)

given NFA $N = (Q, \Sigma, \Delta, S, F)$ with $Q = \{1, \dots, n\}$ and $S = \{1\}$

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- define system of equations

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  with unknowns $X_1, \dots, X_n$

- transform X_1 into regular expression by successive substitution and Arden's lemma

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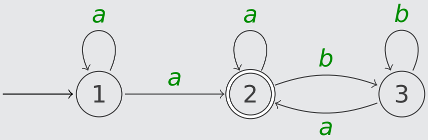

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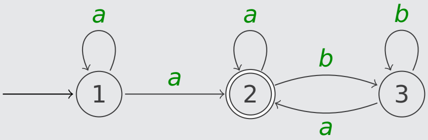

$$X_1 = a\,X_1 + a\,X_2 \qquad X_2 = a\,X_2 + b\,X_3 + \epsilon \qquad X_3 = a\,X_2 + b\,X_3$$

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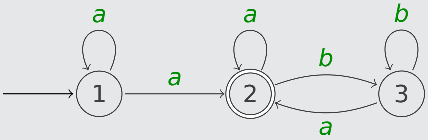

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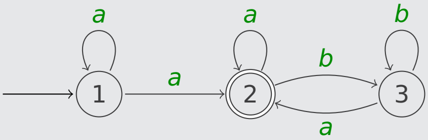
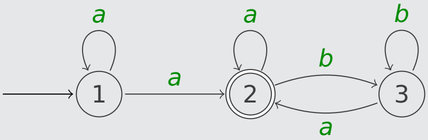
$$X_1 = a^*a\,X_2 \qquad X_2 = a^*(b\,X_3 + \epsilon) \qquad X_3 = b^*a\,X_2 \qquad \text{(Arden's lemma)}$$

Example



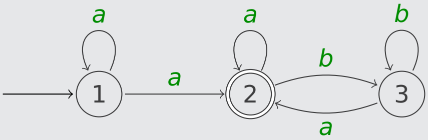
$$
\begin{array}{lll}
X_1 = a\,X_1 + a\,X_2 & X_2 = a\,X_2 + b\,X_3 + \epsilon & X_3 = a\,X_2 + b\,X_3 \\
X_1 = a^* a\,X_2 & X_2 = a^*(b\,X_3 + \epsilon) & X_3 = b^* a\,X_2 \qquad \text{(Arden's lemma)} \\
X_1 = a^* a\,X_2 & X_2 = a^*(bb^* a\,X_2 + \epsilon) & \qquad\qquad \text{(substitute)}
\end{array}
$$

Example



$$
\begin{array}{llll}
X_1 = a X_1 + a X_2 & X_2 = a X_2 + b X_3 + \epsilon & X_3 = a X_2 + b X_3 & \\
X_1 = a^* a X_2 & X_2 = a^*(b X_3 + \epsilon) & X_3 = b^* a X_2 & (\text{Arden's lemma}) \\
X_1 = a^* a X_2 & X_2 = a^*(b b^* a X_2 + \epsilon) & & (\text{substitute}) \\
X_1 = a^* a X_2 & X_2 = a^* b b^* a X_2 + a^* & & (\text{distribute})
\end{array}
$$

Example



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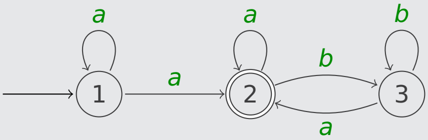
\end{array}$$

Example



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Outline

1. **Summary of Previous Lecture**

2. **Regular Expressions**

3. **Intermezzo**

4. **Homomorphisms**

5. **Decision Problems**

6. **Further Reading**

Particify with session ID **8020 8256**

Question

Which of the following strings belong to $L((aba + ab + b)^*)$?

A ϵ

B $ababa$

C all strings over $\{a, b\}$ that start with b

D all strings over $\{a, b\}$ that do not contain two consecutive b's

Outline

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Theorem

regular sets are effectively closed under homomorphic image and preimage

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Definitions

- homomorphism is mapping $h\colon \Sigma^* \to \Gamma^*$ such that

$$h(\epsilon) = \epsilon \qquad\qquad h(xy) = h(x)h(y)$$

Theorem

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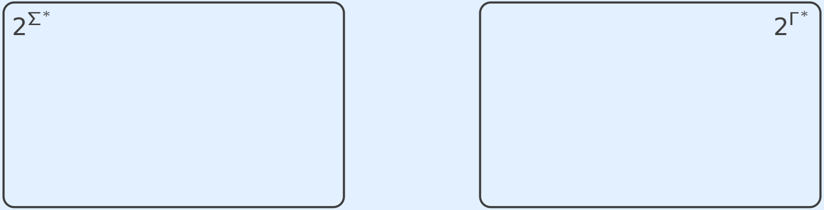
- if $A \subseteq \Sigma^*$ then $h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$ "image of A under h"
- if $B \subseteq \Gamma^*$ then $h^{-1}(B) = \{x \mid h(x) \in B\} \subseteq \Sigma^*$ "preimage of B under h"



2^{Σ^*}

2^{Γ^*}

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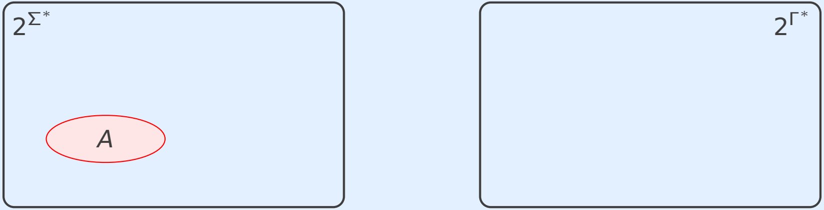

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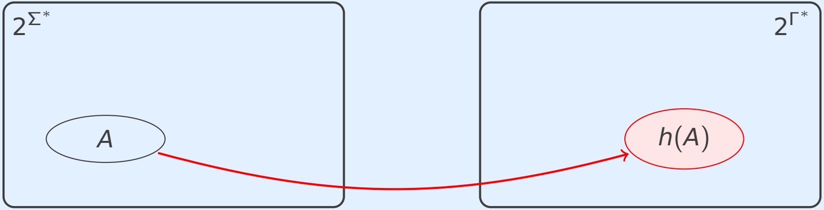

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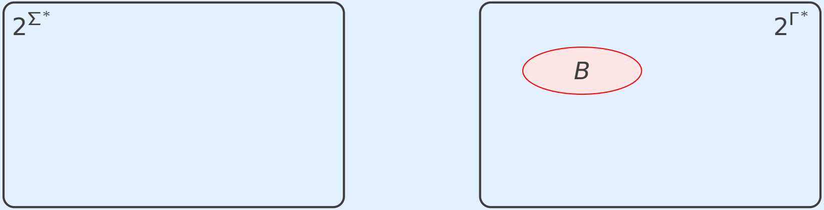

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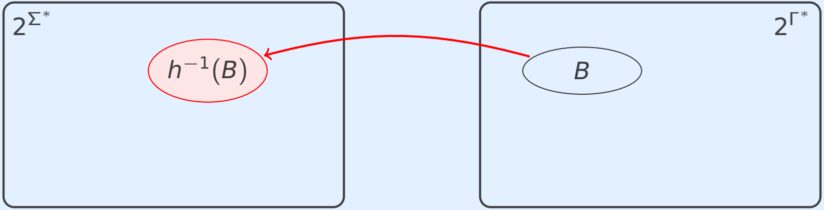

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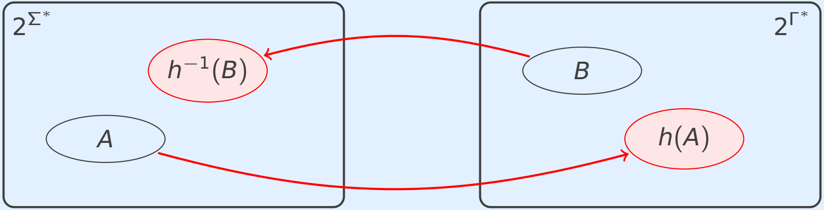

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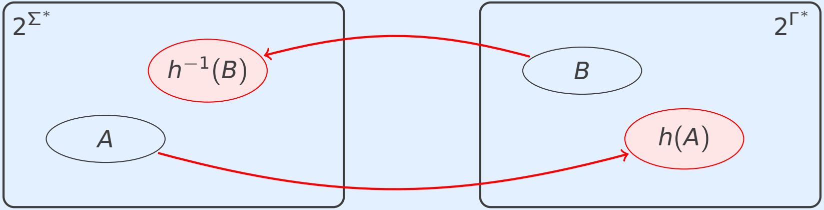

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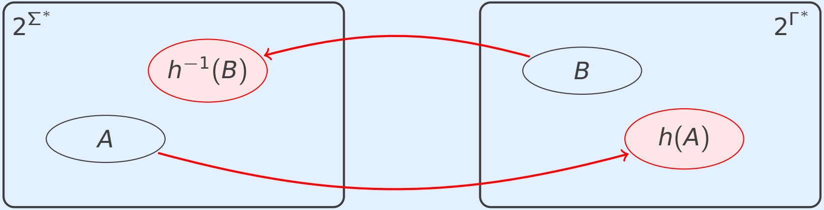
$\Sigma = \Gamma = \{0,1\} \quad h(0) = 11 \quad h(1) = 1$



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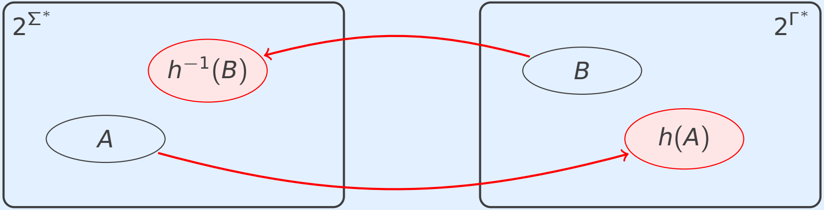

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- $h(h^{-1}(B)) = h(\varnothing) = \varnothing \subsetneq B$

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$A \subseteq \{0,1\}^*$ is regular $\implies$ $\{xy \mid x1y \in A\}$ is regular

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$$h(0) = 0 \qquad h(1) = h(2) = 1 \qquad i(0) = 0 \qquad i(1) = 1 \qquad i(2) = \epsilon$$

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Theorem

regular sets are effectively closed under homomorphic image and preimage

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Proof

- NFA $M = (Q, \Gamma, \Delta, S, F)$
- homomorphism $h \colon \Sigma^* \to \Gamma^*$

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- NFA $M = (Q, \Gamma, \Delta, S, F)$
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- $h^{-1}(L(M)) = L(M')$ for NFA $M' = (Q, \Sigma, \Delta', S, F)$ with $\Delta'(q, a) = \widehat{\Delta}(\{q\}, h(a))$

Theorem

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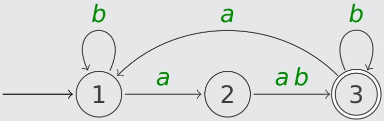
Proof

- NFA $M = (Q, \Gamma, \Delta, S, F)$
- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h^{-1}(L(M)) = L(M')$ for NFA $M' = (Q, \Sigma, \Delta', S, F)$ with $\Delta'(q, a) = \widehat{\Delta}(\{q\}, h(a))$
- claim: $\widehat{\Delta'}(A, x) = \widehat{\Delta}(A, h(x))$ for all $A \subseteq Q$ and $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$

Example

- DFA M



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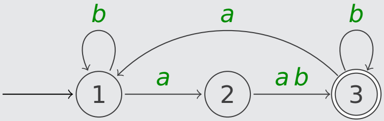

- homomorphism $h\colon \{a, b, c\}^* \to \{a, b\}^*$

$$h(a) = aa \qquad h(b) = \epsilon \qquad h(c) = bab$$

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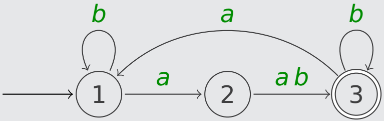

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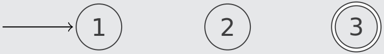

Example

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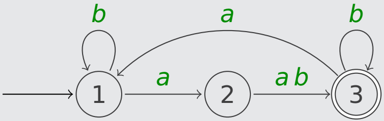

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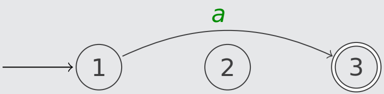

$$\delta'(1,a) = \widehat{\delta}(1,aa) = 3$$

Example

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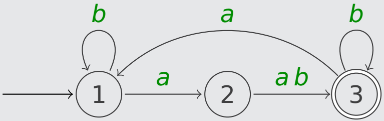

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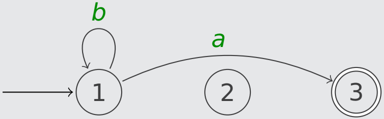

$$\delta'(1, b) = \widehat{\delta}(1, \epsilon) = 1$$

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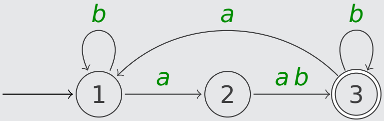

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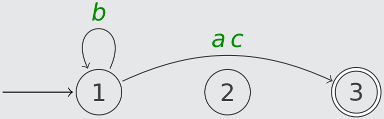

$$\delta'(1,c) = \widehat{\delta}(1,bab) = 3$$

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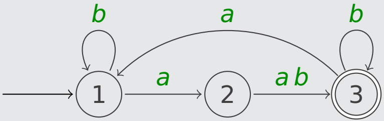

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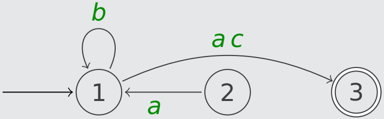

$$\delta'(2, a) = \widehat{\delta}(2, aa) = 1$$

Example

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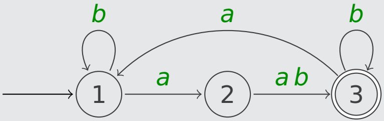

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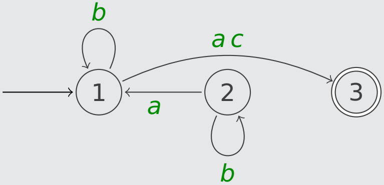

$$\delta'(2,b) = \widehat{\delta}(2,\epsilon) = 2$$

Example

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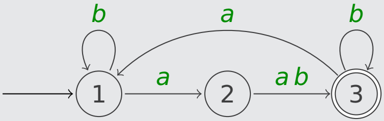

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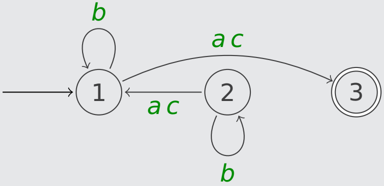

$$\delta'(2,c) = \widehat{\delta}(2,bab) = 1$$

Example

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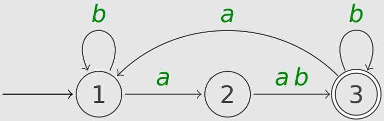

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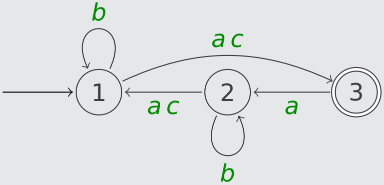

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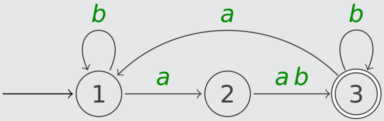

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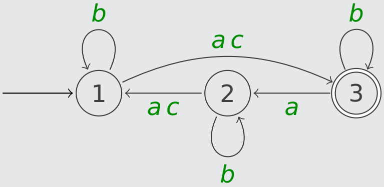

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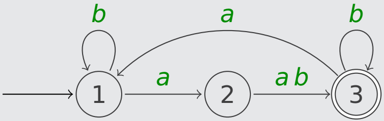
Example

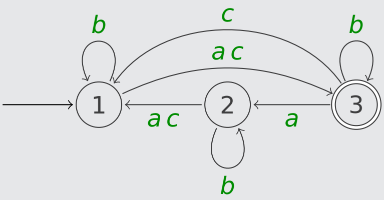
- DFA M



- homomorphism $h\colon \{a,b,c\}^* \to \{a,b\}^*$

$$h(a) = aa \qquad h(b) = \epsilon \qquad h(c) = bab$$

- DFA M'



$$\delta'(3,c) = \widehat{\delta}(3,bab) = 1$$

Theorem

regular sets are effectively closed under homomorphic image and preimage

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Proof

- regular expression α over Σ
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- $h(L(\alpha)) = L(\alpha')$ for regular expression α' defined inductively:

$$a' = h(a) \quad \text{for } a \in \Sigma \qquad\qquad (\beta + \gamma)' = \beta' + \gamma'$$

$$\epsilon' = \epsilon \qquad\qquad (\beta\gamma)' = \beta'\gamma'$$

$$\varnothing' = \varnothing \qquad\qquad {\beta^*}' = {\beta'}^*$$

Definitions

- Hamming distance $H(x,y)$ is number of places where bit strings x and y differ
- if $|x| \neq |y|$ then $H(x,y) = \infty$
- $N_k(A) = \{x \in \{0,1\}^* \mid H(x,y) \leqslant k$ for some $y \in A\}$

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$$D_k = \{x \in (\{0,1\} \times \{0,1\})^* \mid x \text{ contains at most } k \text{ pairs } (0,1) \text{ or } (1,0)\} \qquad \text{is regular}$$

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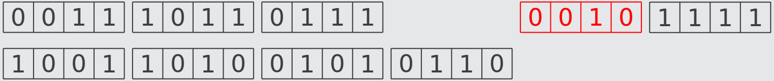

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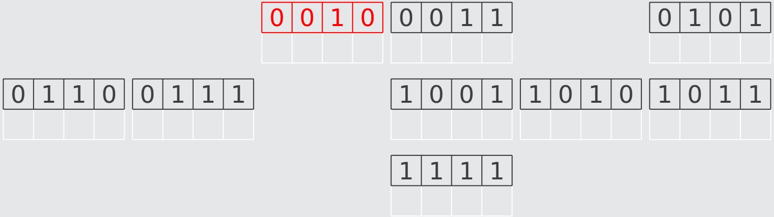

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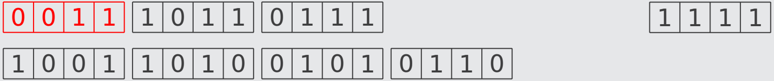

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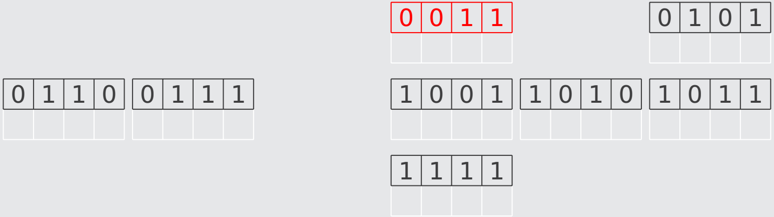

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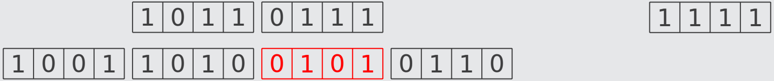

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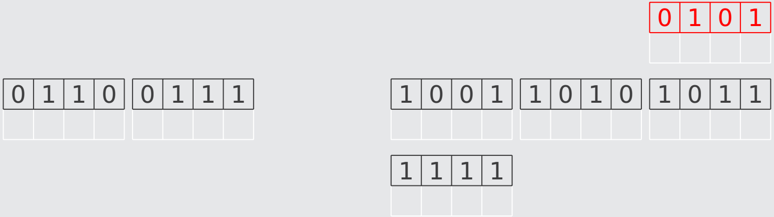

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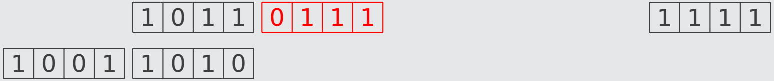

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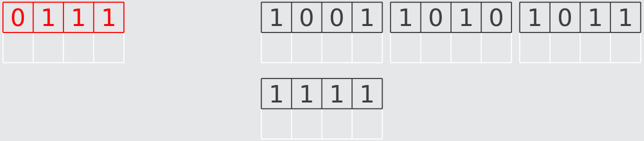

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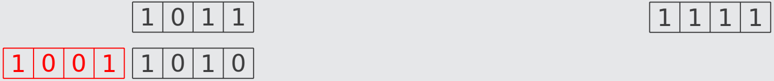

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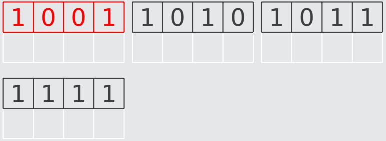

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1. **Summary of Previous Lecture**

2. **Regular Expressions**

3. **Intermezzo**

4. **Homomorphisms**

5. **Decision Problems**

6. **Further Reading**

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question: $x \in L(M)$?

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representation of regular sets (DFA, NFA, regular expression) may affect complexity of decision problems

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Kozen

- Lecture 7 – 10

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homework for October 25