Automata and Logic

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Outline

Definitions

- nondeterministic finite automaton (NFA) is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\Delta \colon Q \times \Sigma \to 2^Q$: transition function
  4. $S \subseteq Q$: set of start states
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\Delta} \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}(A, \epsilon) = A \qquad \widehat{\Delta}(A, xa) = \bigcup_{q \in \widehat{\Delta}(A,x)} \Delta(q, a)$$

- $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

Theorem

every set accepted by NFA is regular

Definitions

- NFA with ϵ-transitions (NFA_ϵ) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that
  1. $\epsilon \notin \Sigma$
  2. $M_\epsilon = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$
- ϵ-closure of set $A \subseteq Q$ is defined as $C_\epsilon(A) = \bigcup \{ \widehat{\Delta}_{N_\epsilon}(A, x) \mid x \in \{\epsilon\}^* \}$
- $\widehat{\Delta}_N \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}_N(A, \epsilon) = C_\epsilon(A) \qquad \widehat{\Delta}_N(A, xa) = \bigcup \{ C_\epsilon(\Delta(q, a)) \mid q \in \widehat{\Delta}_N(A, x) \}$$

Theorem

- every set accepted by NFA_ϵ is regular
- regular sets are effectively closed under union, concatenation, and asterate

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Definitions

- regular expression α over alphabet Σ:

$$a \in \Sigma \qquad \epsilon \qquad \varnothing \qquad \beta + \gamma \qquad \beta\gamma \qquad \beta^*$$

- set of strings $L(\alpha) \subseteq \Sigma^*$ matched by regular expression α:

$$L(a) = \{a\} \qquad L(\beta + \gamma) = L(\beta) \cup L(\gamma)$$
$$L(\epsilon) = \{\epsilon\} \qquad L(\beta\gamma) = L(\beta)L(\gamma)$$
$$L(\varnothing) = \varnothing \qquad L(\beta^*) = L(\beta)^*$$

Example

regular expression $(a + b)^*b$ matches all strings over $\Sigma = \{a, b\}$ that end with b

Definition

regular expressions α and β are equivalent $(\alpha \equiv \beta)$ if $L(\alpha) = L(\beta)$

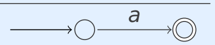
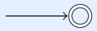
Theorem

finite automata and regular expressions are equivalent:

$$\text{for all } A \subseteq \Sigma^* \quad A \text{ is regular} \iff A = L(\alpha) \text{ for some regular expression } \alpha$$

Proof ($\Longleftarrow$)

induction on regular expression α

α	$L(\alpha)$	finite automaton
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
$\varnothing$	$\varnothing$	

α	$L(\alpha)$
$\beta + \gamma$	$L(\beta) \cup L(\gamma)$
$\beta\gamma$	$L(\beta)L(\gamma)$
β^*	$L(\beta)^*$

$L(\beta)$ and $L(\gamma)$ are regular according to induction hypothesis

$\Longrightarrow$ $L(\alpha)$ is regular according to closure properties of regular sets

Lemma (Arden's Lemma)

if $A, B, X \subseteq \Sigma^*$ such that $X = AX \cup B$ and $\epsilon \notin A$ then $X = A^*B$

Proof

$X \subseteq A^*B$

- let $x \in X = AX \cup B$
- induction on $|x|$
  - $x \in AX \implies x = ay$ for some $a \in A$ and $y \in X \implies y \in A^*B \implies x \in A^*B$
  - $x \in B \implies x \in A^*B$ because $\epsilon \in A^*$

$X \supseteq A^*B$

- $x \in A^*B \implies x = x_1 \cdots x_k y$ for some $x_1, \dots, x_k \in A$ and $y \in B$
- induction on k
  - $k = 0 \implies x = y \in B \subseteq X$
  - $k > 0 \implies x_2 \cdots x_k y \in X \implies x \in AX \subseteq X$

Theorem

finite automata and regular expressions are equivalent:

for all $A \subseteq \Sigma^*$ A is regular $\iff$ $A = L(\alpha)$ for some regular expression α

Proof ($\implies$)

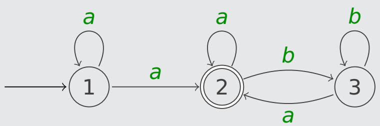
given NFA $N = (Q, \Sigma, \Delta, S, F)$ with $Q = \{1, \dots, n\}$ and $S = \{1\}$

- define system of equations

$$X_i = \left(\bigcup_{a \in \Sigma} \bigcup_{j \in \Delta(i,a)} \{a\} X_j \right) \cup \begin{cases} \{\epsilon\} & \text{if } i \in F \\ \varnothing & \text{otherwise} \end{cases}$$

  with unknowns $X_1, \dots, X_n$
- transform X_1 into regular expression by successive substitution and Arden's lemma

Example



$X_1 = a X_1 + a X_2$ $\quad$ $X_2 = a X_2 + b X_3 + \epsilon$ $\quad$ $X_3 = a X_2 + b X_3$

$X_1 = a^* a X_2$ $\quad$ $X_2 = a^*(b X_3 + \epsilon)$ $\quad$ $X_3 = b^* a X_2$ (Arden's lemma)

$X_1 = a^* a X_2$ $\quad$ $X_2 = a^*(bb^* a X_2 + \epsilon)$ (substitute)

$X_1 = a^* a X_2$ $\quad$ $X_2 = a^* bb^* a X_2 + a^*$ (distribute)

$X_1 = a^* a X_2$ $\quad$ $X_2 = (a^* bb^* a)^* a^*$ (Arden's lemma)

$X_1 = a^* a (a^* bb^* a)^* a^*$ (substitute)

Outline

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Question

Which of the following strings belong to $L((aba + ab + b)^*)$?

A ϵ

B $ababa$

C all strings over $\{a, b\}$ that start with b

D all strings over $\{a, b\}$ that do not contain two consecutive b's

Outline

Theorem

regular sets are effectively closed under homomorphic image and preimage

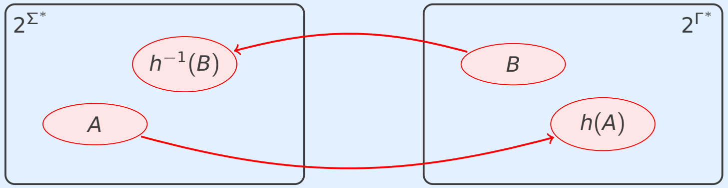
Definitions

- homomorphism is mapping $h\colon \Sigma^* \to \Gamma^*$ such that

$$h(\epsilon) = \epsilon \qquad\qquad h(xy) = h(x)h(y)$$

so homomorphism is completely determined by its effect on Σ

- if $A \subseteq \Sigma^*$ then $h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$ "image of A under h"
- if $B \subseteq \Gamma^*$ then $h^{-1}(B) = \{x \mid h(x) \in B\} \subseteq \Sigma^*$ "preimage of B under h"



- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h^{-1}(h(A)) \supseteq A$
- $h(h^{-1}(B)) \subseteq B$

Example

$\Sigma = \Gamma = \{0, 1\} \quad h(0) = 11 \quad h(1) = 1 \quad A = B = \{0\}$

- $h^{-1}(h(A)) = h^{-1}(\{11\}) = \{0, 11\} \supsetneq A$
- $h(h^{-1}(B)) = h(\varnothing) = \varnothing \subsetneq B$

Example

$A \subseteq \{0,1\}^*$ is regular $\quad\Longrightarrow\quad$ $\{xy \mid x1y \in A\}$ is regular

- $\Sigma = \{0,1\}$ and $\Gamma = \{0,1,2\}$
- define homomorphisms $h, i\colon \Gamma^* \to \Sigma^*$ by

$$h(0) = 0 \qquad h(1) = h(2) = 1 \qquad i(0) = 0 \qquad i(1) = 1 \qquad i(2) = \epsilon$$

- $h^{-1}(A) = \{x \mid h(x) \in A\}$
- $h^{-1}(A) \cap L((0+1)^*2(0+1)^*) = \{x2y \mid x1y \in A\}$
- $\{xy \mid x1y \in A\} = i(h^{-1}(A) \cap L((0+1)^*2(0+1)^*))$ is regular

Theorem

regular sets are effectively closed under homomorphic image and preimage

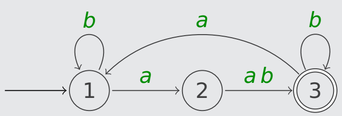
Proof

- NFA $M = (Q, \Gamma, \Delta, S, F)$
- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h^{-1}(L(M)) = L(M')$ for NFA $M' = (Q, \Sigma, \Delta', S, F)$ with $\Delta'(q,a) = \widehat{\Delta}(\{q\}, h(a))$
- claim: $\quad \widehat{\Delta'}(A,x) = \widehat{\Delta}(A, h(x)) \quad$ for all $A \subseteq Q$ and $x \in \Sigma^*$

  proof of claim: $\quad$ easy induction on $|x|$

Example

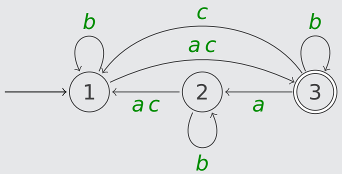
- DFA M



- homomorphism $h\colon \{a,b,c\}^* \to \{a,b\}^*$

$$h(a) = aa \qquad h(b) = \epsilon \qquad h(c) = bab$$

- DFA M'



$\delta'(3,c) = \widehat{\delta}(3, bab) = 1$

Theorem

regular sets are effectively closed under homomorphic image and preimage

Proof

- regular expression α over Σ
- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h(L(\alpha)) = L(\alpha')$ for regular expression α' defined inductively:

$$a' = h(a) \quad \text{for } a \in \Sigma \qquad\qquad (\beta+\gamma)' = \beta' + \gamma'$$
$$\epsilon' = \epsilon \qquad\qquad (\beta\gamma)' = \beta'\gamma'$$
$$\varnothing' = \varnothing \qquad\qquad \beta^{*\prime} = \beta'^{*}$$

Definitions

- Hamming distance $H(x,y)$ is number of places where bit strings x and y differ
- if $|x| \neq |y|$ then $H(x,y) = \infty$
- $N_k(A) = \{x \in \{0,1\}^* \mid H(x,y) \leqslant k \text{ for some } y \in A\}$

Lemma

$A \subseteq \{0,1\}^*$ is regular $\implies$ $\forall\, k \in \mathbb{N}$ $N_k(A)$ is regular

Proof

$$D_k = \{x \in (\{0,1\} \times \{0,1\})^* \mid x \text{ contains at most } k \text{ pairs } (0,1) \text{ or } (1,0)\} \quad \text{is regular}$$
$$= \{x \in (\{0,1\} \times \{0,1\})^* \mid H(\mathsf{fst}(x), \mathsf{snd}(x)) \leqslant k\}$$

$$N_k(A) = \mathsf{fst}(\mathsf{snd}^{-1}(A) \cap D_k)$$

Example

- $A = \{0011\}$ $k = 2$
- $N_k(A)$ consists of

0011 1011 0111 0001 0010 1111
1001 1010 0101 0110 0000

- $\mathsf{fst}(\mathsf{snd}^{-1}(A) \cap D_k)$ consists of

0000 / 0011	0001 / 0011	0010 / 0011	0011 / 0011	0100 / 0011	0101 / 0011
0110 / 0011	0111 / 0011	1000 / 0011	1001 / 0011	1010 / 0011	1011 / 0011
1100 / 0011	1101 / 0011	1110 / 0011	1111 / 0011		

Outline

Remark

most decision problems concerning regular sets are decidable

Theorem

problems

instance: DFA M and string x	instance: DFA M	instance: DFAs M and N
question: $x \in L(M)$?	question: $L(M) = \varnothing$?	question: $L(M) = L(N)$?

are decidable

Remark

representation of regular sets (DFA, NFA, regular expression) may affect complexity of decision problems

Outline

Kozen

- Lecture 7 – 10

Important Concepts

- Arden's lemma
- homomorphism
- homomorphic image
- homomorphic preimage
- regular expression

homework for October 25