Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

1. **Summary of Previous Lecture**
2. **Minimization**
3. **Intermezzo**
4. **Weak Monadic Second-Order Logic**
5. **Further Reading**

Definitions

- regular expression α over alphabet Σ:

$$a \in \Sigma \qquad \epsilon \qquad \varnothing \qquad \beta + \gamma \qquad \beta\gamma \qquad \beta^*$$

- set of strings $L(\alpha) \subseteq \Sigma^*$ matched by regular expression α:

$$L(a) = \{a\} \qquad L(\varnothing) = \varnothing \qquad L(\beta\gamma) = L(\beta)L(\gamma)$$
$$L(\epsilon) = \{\epsilon\} \qquad L(\beta + \gamma) = L(\beta) \cup L(\gamma) \qquad L(\beta^*) = L(\beta)^*$$

- regular expressions α and β are equivalent $(\alpha \equiv \beta)$ if $L(\alpha) = L(\beta)$

Theorem

finite automata and regular expressions are equivalent:

$$\text{for all } A \subseteq \Sigma^* \quad A \text{ is regular} \iff A = L(\alpha) \text{ for some regular expression } \alpha$$

Definitions

- homomorphism is mapping $h\colon \Sigma^* \to \Gamma^*$ such that $h(\epsilon) = \epsilon$ and $h(xy) = h(x)h(y)$
- if $A \subseteq \Sigma^*$ then $h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$ "image of A under h"
- if $B \subseteq \Gamma^*$ then $h^{-1}(B) = \{x \mid h(x) \in B\} \subseteq \Sigma^*$ "preimage of B under h"

Theorem

regular sets are effectively closed under homomorphic image and preimage

Theorem

problems

instance: DFA M and string x	instance: DFA M	instance: DFAs M and N
question: $x \in L(M)$?	question: $L(M) = \varnothing$?	question: $L(M) = L(N)$?

are decidable

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

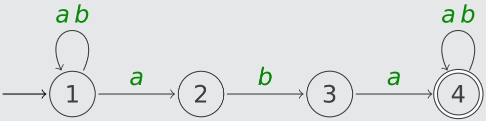
Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Example ❶

NFA



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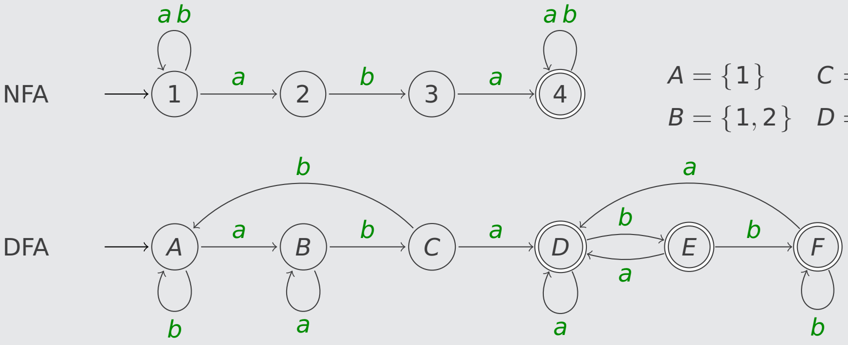

$A = \{1\}$ $\quad C = \{1,3\}$ $\quad E = \{1,3,4\}$

$B = \{1,2\}$ $\quad D = \{1,2,4\}$ $\quad F = \{1,4\}$

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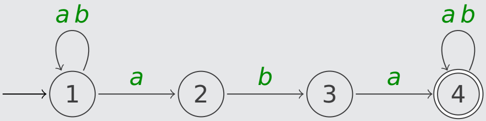

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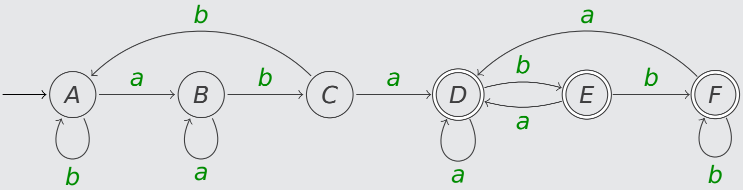

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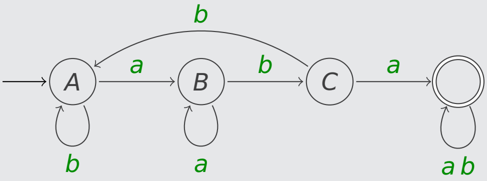

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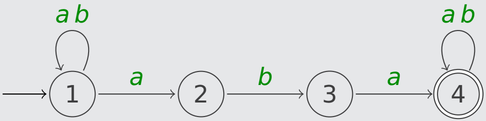

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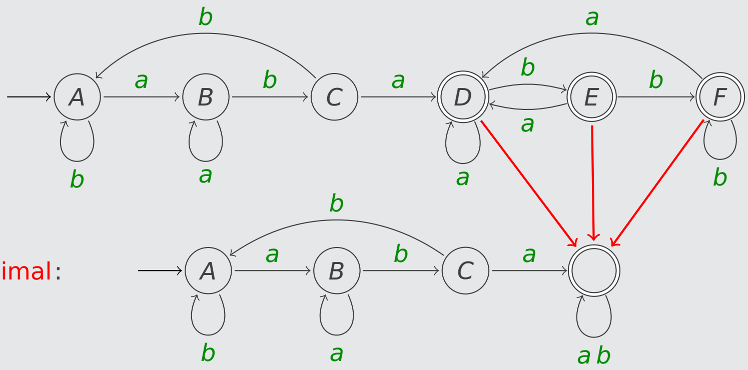

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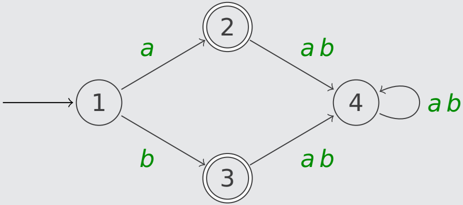

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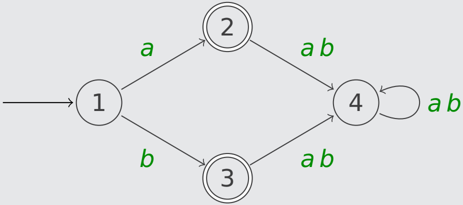

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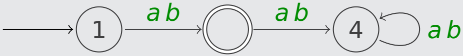

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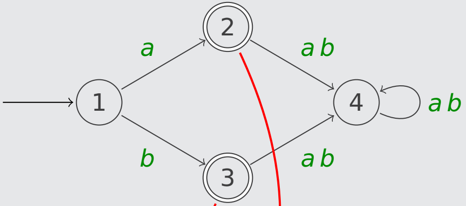

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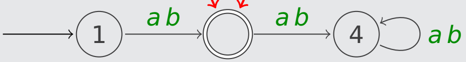

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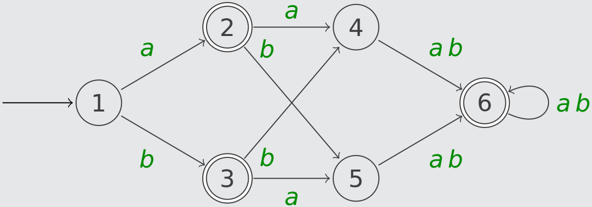

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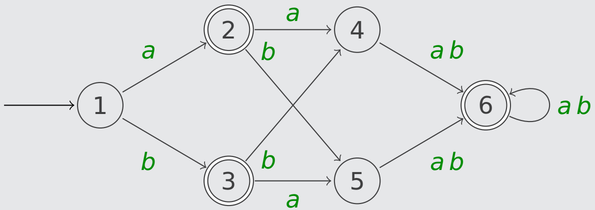

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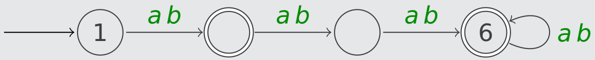

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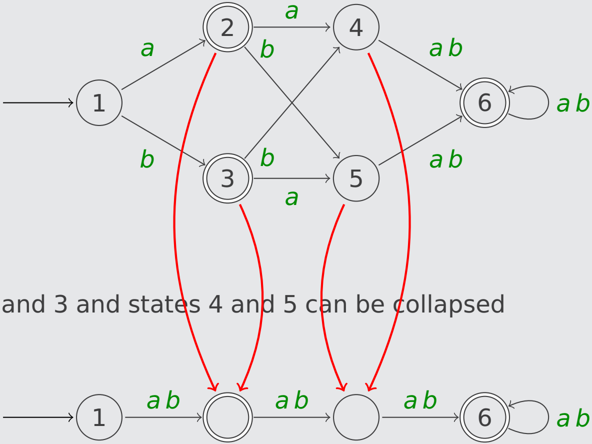

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③ collapse indistinguishable states

Marking Algorithm

given DFA $M = (Q, \Sigma, \delta, s, F)$ without inaccessible states

① tabulate all unordered pairs $\{p, q\}$ with $p, q \in Q$, initially unmarked

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Notation

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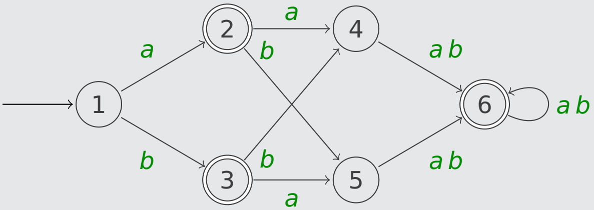
Lemma

$p \approx q \iff \{p, q\}$ is unmarked

Example



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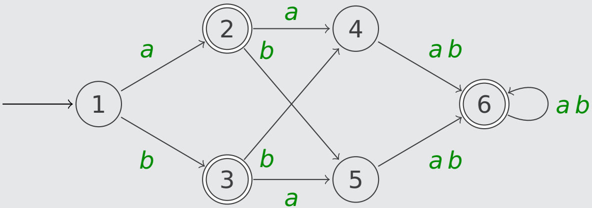

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1. final / non-final states are distinguishable

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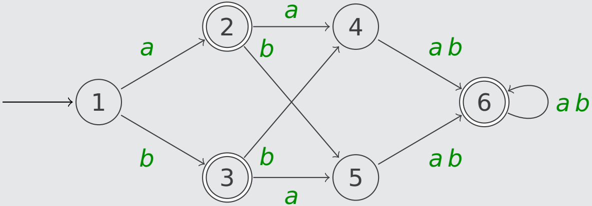

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2. $\{2,6\} \xrightarrow{a} \{4,6\}$ $\{3,6\} \xrightarrow{a} \{5,6\}$

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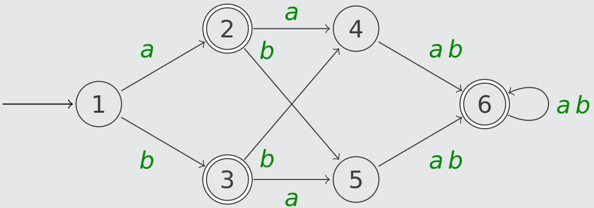

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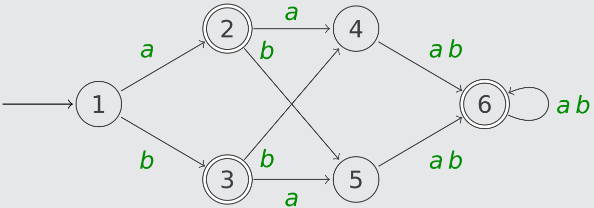

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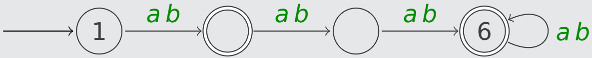

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Notation

$[p]_{\approx} = \{q \in Q \mid p \approx q\}$ denotes equivalence class of p

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Theorem

$L(M/\approx) = L(M)$

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is $M/\approx$ minimum-state DFA for $L(M)$?

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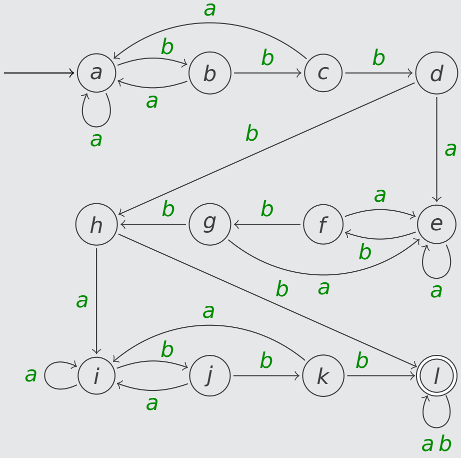
Lemma

$M/{\approx}$ cannot be collapsed further

Example

DFA for set of strings over $\{a, b\}$ containing at least three occurrences of three consecutive b's, overlapping permitted:



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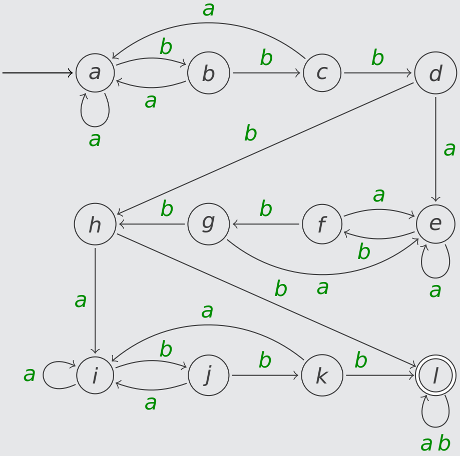

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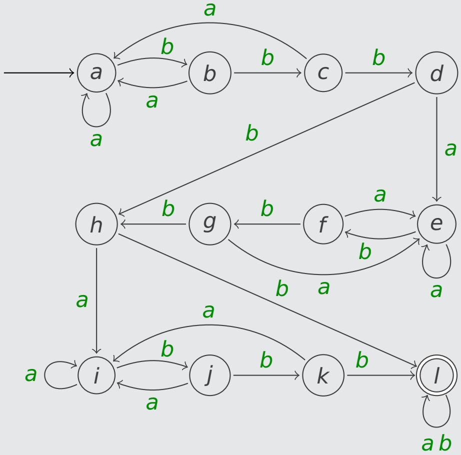

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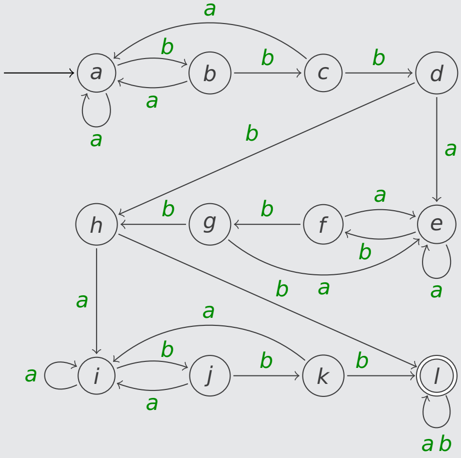

	a	b	c	d	e	f	g	h	i	j	k
b											
c											
d	✓	✓	✓								
e				✓							
f				✓							
g	✓	✓	✓		✓	✓					
h	✓	✓	✓	✓	✓	✓	✓				
i				✓			✓	✓			
j	✓	✓	✓		✓	✓		✓	✓		
k	✓	✓	✓	✓	✓	✓	✓		✓	✓	
l	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

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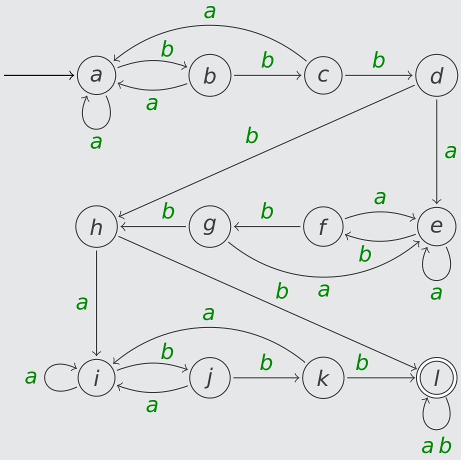

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b											
c	✓	✓									
d	✓	✓	✓								
e			✓	✓							
f	✓	✓		✓	✓						
g	✓	✓	✓		✓	✓					
h	✓	✓	✓	✓	✓	✓	✓				
i	✓	✓		✓	✓		✓	✓			
j	✓	✓	✓		✓	✓		✓	✓		
k	✓	✓	✓	✓	✓	✓	✓		✓	✓	
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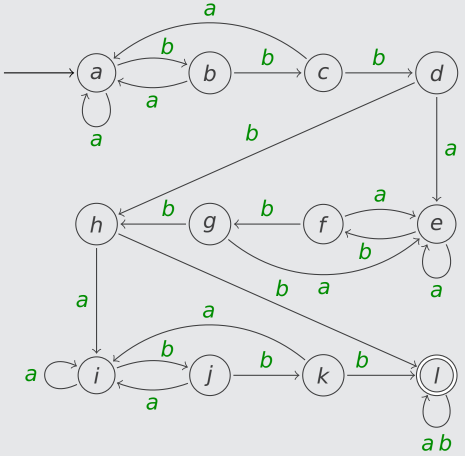

	a	b	c	d	e	f	g	h	i	j	k
b	√										
c	√	√									
d	√	√	√								
e	√		√	√							
f	√	√		√	√						
g	√	√	√		√	√					
h	√	√	√	√	√	√	√				
i	√	√	√	√	√	√	√	√			
j	√	√	√	√	√	√	√	√	√		
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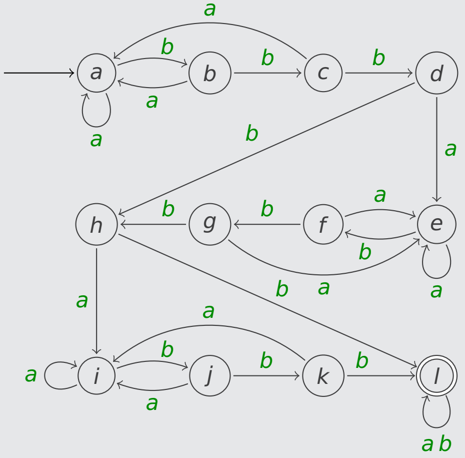

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f	✓	✓	✓	✓	✓						
g	✓	✓	✓		✓	✓					
h	✓	✓	✓	✓	✓	✓	✓				
i	✓	✓	✓	✓	✓	✓	✓	✓			
j	✓	✓	✓	✓	✓	✓	✓	✓	✓		
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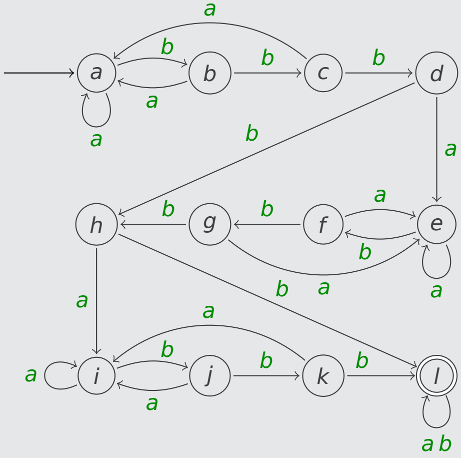

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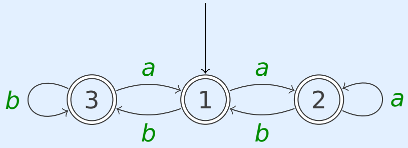
✓	✓	✓	✓	✓	f						
✓	✓	✓		✓	✓	g					
✓	✓	✓	✓	✓	✓	✓	h				
✓	✓	✓	✓	✓	✓	✓	✓	i			
✓	✓	✓	✓	✓	✓	✓	✓	✓	j		
✓	✓	✓	✓	✓	✓	✓		✓	✓	k	
✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	l

states d, g and h, k can be merged

Outline

Question

Which statements about the following DFA are true ?



A states 2 and 3 are distinguishable

B all states can be merged

C the DFA is minimal

D states 1 and 2 can be merged

Outline

Definitions

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Remark

$z = y + 1$ abbreviates $y < z \wedge \neg\exists\, x.\, (y < x \wedge x < z)$

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\end{aligned}
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- size of model α is smallest n such that
  1. $\alpha(x) < n$ for $x \in V_1$
  2. $\alpha(X) \subseteq \{0, \dots, n-1\}$ for $X \in V_2$

Examples

- $(\forall x.\, X(x) \to Y(x)) \wedge (\exists y.\, \neg X(y) \wedge Y(y))$ — satisfiable
- $(\forall x.\, x = 0 \to X(x)) \wedge (\forall x.\, X(x) \to \exists y.\, x < y \wedge X(y))$ — unsatisfiable
- $(\exists x.\, X(x) \wedge \exists y.\, X(y) \wedge x \neq y) \wedge (\forall x.\, X(x) \to \exists y.\, Y(y) \wedge x < y)$ — satisfiable

Definition

given alphabet Σ

- second-order variables $V_2 = \{P_a \mid a \in \Sigma\}$

Definition

given alphabet Σ and string $x = a_0 \cdots a_{n-1} \in \Sigma^*$

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$\Sigma = \{a, b\}$

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- $\underline{abba}(P_b) = \{1, 2\}$

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$\Sigma = \{a, b\}$

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- $\varphi = \forall x.\, \neg(P_a(x) \wedge P_b(x))$ $\qquad$ $\underline{x} \vDash \varphi$ for all $x \in \Sigma^*$

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Definitions

- given alphabet Σ and WMSO formula φ with free variables (exclusively) in $\{P_a \mid a \in \Sigma\}$

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$$L(\varphi) = \{x \in \Sigma^* \mid \underline{x} \vDash \varphi\}$$

- set $A \subseteq \Sigma^*$ is WMSO definable if $A = L(\varphi)$ for some WMSO formula φ

Examples

$\Sigma = \{a, b\}$

- regular set $L((a+b)^*ab(a+b)^*)$ is WMSO definable by formula

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Theorem

set $A \subseteq \Sigma^*$ is regular if and only if A is WMSO definable

Outline

1. **Summary of Previous Lecture**
2. **Minimization**
3. **Intermezzo**
4. **Weak Monadic Second-Order Logic**
5. **Further Reading**

Kozen

- Lectures 13 and 14

Kozen

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Important Concepts

- $\alpha \vDash \varphi$
- indistinguishable states
- minimization algorithm
- model
- MSO
- satisfiability
- validity
- weak monadic second-order logic (WMSO)
- WMSO definability

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homework for November 8