WS 2024 lecture 6

Automata and Logic

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Outline

Definition

equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

$$x \equiv_M y \iff \widehat{\delta}(s, x) = \widehat{\delta}(s, y)$$

Lemmata

- $\equiv_M$ is right congruent: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ for all $a \in \Sigma$ $xa \equiv_M ya$
- $\equiv_M$ refines $L(M)$: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ either $x, y \in L(M)$ or $x, y \notin L(M)$
- $\equiv_M$ is of finite index: $\equiv_M$ has finitely many equivalence classes

Definition

Myhill–Nerode relation for $L \subseteq \Sigma^*$ is right congruent equivalence relation of finite index on Σ^* that refines L

Definition

for any set $L \subseteq \Sigma^*$ equivalence relation $\equiv_L$ on Σ^* is defined as follows:

$$x \equiv_L y \iff \text{for all } z \in \Sigma^* \ \left(xz \in L \iff yz \in L \right)$$

Theorem

1. for every regular set $L \subseteq \Sigma^*$

   there exists one-to-one correspondence (up to isomorphism of automata) between

   - DFAs for L with input alphabet Σ and without inaccessible states
   - Myhill–Nerode relations for L

2. for every set $L \subseteq \Sigma^*$

   $$L \text{ is regular} \iff L \text{ admits Myhill–Nerode relation} \iff \equiv_L \text{ is of finite index}$$

3. regular sets are WMSO definable

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Definitions

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$\varphi = \exists X.\, X(x) \rightarrow \exists y.\, x < y \wedge Y(y)$

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Definitions

- $\mathrm{FV}(\varphi)$ denotes list of free variables in φ in fixed order with first-order variables preceding second-order ones
- assignment for φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is tuple $(i_1, \dots, i_m, I_1, \dots, I_n)$ such that $i_1, \dots, i_m$ are elements of $\mathbb{N}$ and $I_1, \dots, I_n$ are finite subsets of $\mathbb{N}$

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Notation

$(\boldsymbol{i}, \boldsymbol{I})$ for $(i_1, \dots, i_m, I_1, \dots, I_n)$ $\qquad$ $\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Example

x_1	0	0	0	1	0	0
X_1	1	0	1	0	1	0
X_2	0	0	0	0	0	0

Example

	0	1	2	3	4	5
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induced assignment

$i_1 = 3$

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Example

	0	1	2	3	4	5	induced assignment
x_1	0	0	0	1	0	0	$i_1 = 3$
X_1	1	0	1	0	1	0	$I_1 = \{0, 2, 4\}$
X_2	0	0	0	0	0	0	$I_2 = \varnothing$

Example

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induced assignment

$i_1 = 3$

$I_1 = \{0,2,4\}$

$I_2 = \varnothing$

Remark

assignments are identified with strings over $\{0,1\}^{m+n}$ (here $k = m + n$)

$$\begin{matrix} a_{11} & a_{21} & \cdots & a_{\ell 1} \\ \vdots & \vdots & & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{\ell k} \end{matrix} \approx \begin{pmatrix} a_{11} \\ \vdots \\ a_{1k} \end{pmatrix} \begin{pmatrix} a_{21} \\ \vdots \\ a_{2k} \end{pmatrix} \cdots \begin{pmatrix} a_{\ell 1} \\ \vdots \\ a_{\ell k} \end{pmatrix}$$

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string over $\{0,1\}^{m+n}$ is *m-admissible* if first m rows contain exactly one 1 each

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- if $x, y \in (\{0,1\}^{m+n})^*$ induce same assignment then $x = y\mathbf{0}\cdots\mathbf{0}$ or $y = x\mathbf{0}\cdots\mathbf{0}$

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- $\epsilon \in (\{0,1\}^{m+n})^*$ is m-admissible if and only if $m = 0$
- if $\underline{x} = (\boldsymbol{i}, \boldsymbol{I})$ then $|x| > k$ for all $k \in \{i_1, \dots, i_m\} \cup I_1 \cup \cdots \cup I_n$

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Lemma

set of m-admissible strings over $\{0,1\}^{m+n}$ is regular

Outline

Question

Consider the following three strings over $\{0,1\}^{1+2}$. Which statements hold ?

	0	1	2
x_1	0	1	0
X_1	1	0	0
X_2	0	1	1

	0	1	2
x_1	0	0	0
X_1	1	0	0
X_2	0	1	1

	0	1	2	3
x_1	0	1	0	0
X_1	1	0	0	0
X_2	0	1	1	0

A the first and third string induce the same assignment

B the second string is 1-admissible

C $X_2(x_1) \rightarrow X_1(x_1)$ is satisfied by the first string's induced assignment

D the third string induces the assignment $i_1 = 1$, $I_1 = \{0\}$, $I_2 = \{1,2\}$

Outline

Lemma

set of m-admissible strings over $\{0,1\}^{m+n}$ is regular

Proof (construction)

define DFA $\mathcal{A}_{m,n} = (Q, \Sigma, \delta, s, F)$

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⑤ $$\delta(q,a) = \begin{cases} q - I & \text{if } q \subseteq \{1,\dots,m\} \text{ and } I \subseteq q \\ \bot & \text{if } q \subseteq \{1,\dots,m\} \text{ and } I \not\subseteq q \\ \bot & \text{if } q = \bot \end{cases}$$

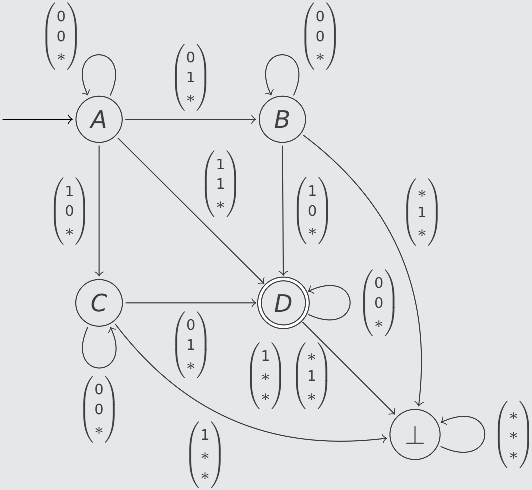
with $I = \{i \in \{1,\dots,m\} \mid a_i = 1\}$

Example

DFA $\mathcal{A}_{2,1}$

Example

DFA $\mathcal{A}_{2,1}$ with $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$, $D = \varnothing$



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- $\varphi(x, X) = \forall\, y.\, y < x \to X(y)$
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Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

Proof

induction on φ

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- $\varphi = \bot$
- $\varphi = x < y$
- $\varphi = X(x)$
- $\varphi = \neg\psi$
- $\varphi = \varphi_1 \vee \varphi_2$

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- $\varphi = X(x) \quad \Longrightarrow \quad L_a(\varphi) = \left[\binom{0}{0} + \binom{0}{1}\right]^* \binom{1}{1} \left[\binom{0}{0} + \binom{0}{1}\right]^*$
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- $\varphi = \neg\psi \quad \Longrightarrow \quad L_a(\psi)$ is regular
- $\varphi = \varphi_1 \vee \varphi_2$

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- $\varphi = \varphi_1 \vee \varphi_2$

  $L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

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Definition

homomorphism

$$\text{drop}_i \colon (\{0,1\}^k)^* \to (\{0,1\}^{k-1})^*$$

is defined for $1 \leqslant i \leqslant k$ by dropping i-th component from vectors in $\{0,1\}^k$

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Lemmata

- $A \subseteq (\{0,1\}^k)^*$ is regular $\implies$ $\mathrm{drop}_i(A) \subseteq (\{0,1\}^{k-1})^*$ is regular

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- $B \subseteq (\{0,1\}^{k-1})^*$ is regular $\implies$ $\text{drop}_i^{-1}(B) \subseteq (\{0,1\}^k)^*$ is regular

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- $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{2,1})$

Proof

induction on φ

- $\varphi = \bot \implies L_a(\varphi) = \varnothing$
- $\varphi = x < y \implies L_a(\varphi) = \binom{0}{0}^*\binom{1}{0}\binom{0}{0}^*\binom{0}{1}\binom{0}{0}^*$ or $L_a(\varphi) = \binom{0}{0}^*\binom{0}{1}\binom{0}{0}^*\binom{1}{0}\binom{0}{0}^*$
- $\varphi = X(x) \implies L_a(\varphi) = \left[\binom{0}{0} + \binom{0}{1}\right]^*\binom{1}{1}\left[\binom{0}{0} + \binom{0}{1}\right]^*$
- $\varphi = \neg\psi \implies L_a(\psi)$ is regular $\implies \sim L_a(\psi)$ is regular

  $\implies L_a(\varphi) = \sim L_a(\psi) \cap L(\mathcal{A}_{m,n})$ is regular (for suitable m and n)
- $\varphi = \varphi_1 \vee \varphi_2$ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$

  $L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

  applications of inverse homomorphism drop_i^{-1} to $L_a(\varphi_1)$ and $L_a(\varphi_2)$ yield regular sets $L_1, L_2 \subseteq (\{0,1\}^{m+n})^*$ such that $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{m,n})$

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$\varphi = \exists X.\, \psi$ with $\psi = X(x) \wedge \exists y.\, x < y \wedge X(y)$

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Definition

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transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$

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Corollary

WMSO definable sets are regular

Outline

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Demo

`https://www.brics.dk/mona/`

Outline

Ebbinghaus, Flum and Thomas

- Section 10.9 of Einführung in die mathematische Logik (Springer Spektrum 2018)

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Important Concepts

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- m-admissible
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homework for November 15