universität innsbruck

WS 2024 lecture 6

Automata and Logic

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Outline

Definition

equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

$$x \equiv_M y \iff \widehat{\delta}(s,x) = \widehat{\delta}(s,y)$$

Lemmata

- $\equiv_M$ is right congruent: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ for all $a \in \Sigma$ $xa \equiv_M ya$
- $\equiv_M$ refines $L(M)$: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ either $x, y \in L(M)$ or $x, y \notin L(M)$
- $\equiv_M$ is of finite index: $\equiv_M$ has finitely many equivalence classes

Definition

Myhill–Nerode relation for $L \subseteq \Sigma^*$ is right congruent equivalence relation of finite index on Σ^* that refines L

Definition

for any set $L \subseteq \Sigma^*$ equivalence relation $\equiv_L$ on Σ^* is defined as follows:

$$x \equiv_L y \iff \text{for all } z \in \Sigma^* \ \big(xz \in L \iff yz \in L\big)$$

Theorem

1. for every regular set $L \subseteq \Sigma^*$
   there exists one-to-one correspondence (up to isomorphism of automata) between
   - DFAs for L with input alphabet Σ and without inaccessible states
   - Myhill–Nerode relations for L
2. for every set $L \subseteq \Sigma^*$
   L is regular $\iff$ L admits Myhill–Nerode relation $\iff$ $\equiv_L$ is of finite index
3. regular sets are WMSO definable

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Definitions

- $\mathrm{FV}(\varphi)$ denotes list of free variables in φ in fixed order with first-order variables preceding second-order ones
- assignment for φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is tuple $(i_1, \dots, i_m, I_1, \dots, I_n)$ such that $i_1, \dots, i_m$ are elements of $\mathbb{N}$ and $I_1, \dots, I_n$ are finite subsets of $\mathbb{N}$

Example

$\varphi = \exists X.\, X(x) \to \exists y.\, x < y \wedge Y(y)$ $\qquad \mathrm{FV}(\varphi) = (x, Y)$

Notation

$(\boldsymbol{i}, \boldsymbol{I})$ for $(i_1, \dots, i_m, I_1, \dots, I_n)$ $\qquad \mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Example

	0	1	2	3	4	5	induced assignment
x_1	0	0	0	1	0	0	$i_1 = 3$
X_1	1	0	1	0	1	0	$I_1 = \{0, 2, 4\}$
X_2	0	0	0	0	0	0	$I_2 = \varnothing$

Remark

assignments are identified with strings over $\{0,1\}^{m+n}$ (here $k = m + n$)

$$\begin{matrix} a_{11} & a_{21} & \cdots & a_{\ell 1} \\ \vdots & \vdots & & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{\ell k} \end{matrix} \approx \begin{pmatrix} a_{11} \\ \vdots \\ a_{1k} \end{pmatrix} \begin{pmatrix} a_{21} \\ \vdots \\ a_{2k} \end{pmatrix} \cdots \begin{pmatrix} a_{\ell 1} \\ \vdots \\ a_{\ell k} \end{pmatrix}$$

Definition

string over $\{0,1\}^{m+n}$ is *m-admissible* if first m rows contain exactly one 1 each

Remarks

- every m-admissible string x induces assignment $\underline{x}$
- every assignment is induced by (not necessarily unique) m-admissible string: if x is m-admissible then $x\mathbf{0}$ is m-admissible and $\underline{x} = \underline{x\mathbf{0}}$
- if $x, y \in (\{0,1\}^{m+n})^*$ induce same assignment then $x = y\mathbf{0}\cdots\mathbf{0}$ or $y = x\mathbf{0}\cdots\mathbf{0}$
- $\epsilon \in (\{0,1\}^{m+n})^*$ is m-admissible if and only if $m = 0$
- if $\underline{x} = (\boldsymbol{i}, \boldsymbol{I})$ then $|x| > k$ for all $k \in \{i_1, \dots, i_m\} \cup I_1 \cup \cdots \cup I_n$

Lemma

set of m-admissible strings over $\{0,1\}^{m+n}$ is regular

Outline

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Question

Consider the following three strings over $\{0,1\}^{1+2}$. Which statements hold ?

	0	1	2
x_1	0	1	0
X_1	1	0	0
X_2	0	1	1

	0	1	2
x_1	0	0	0
X_1	1	0	0
X_2	0	1	1

	0	1	2	3
x_1	0	1	0	0
X_1	1	0	0	0
X_2	0	1	1	0

A the first and third string induce the same assignment

B the second string is 1-admissible

C $X_2(x_1) \to X_1(x_1)$ is satisfied by the first string's induced assignment

D the third string induces the assignment $i_1 = 1,\ I_1 = \{0\},\ I_2 = \{1,2\}$

Outline

Lemma

set of m-admissible strings over $\{0,1\}^{m+n}$ is regular

Proof (construction)

define DFA $\mathcal{A}_{m,n} = (Q, \Sigma, \delta, s, F)$ with

① $\Sigma = \{0,1\}^{m+n}$

② $Q = 2^{\{1,\dots,m\}} \cup \{\bot\}$

③ $s = \{1,\dots,m\}$

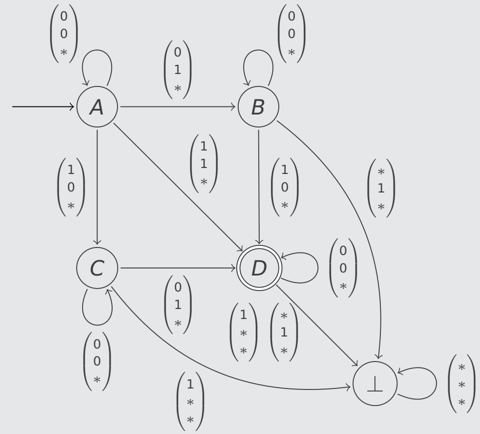
④ $F = \{\varnothing\}$

⑤ $\delta(q,a) = \begin{cases} q - I & \text{if } q \subseteq \{1,\dots,m\} \text{ and } I \subseteq q \\ \bot & \text{if } q \subseteq \{1,\dots,m\} \text{ and } I \nsubseteq q \\ \bot & \text{if } q = \bot \end{cases}$

with $I = \{i \in \{1,\dots,m\} \mid a_i = 1\}$

Example

DFA $\mathcal{A}_{2,1}$ with $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$, $D = \varnothing$



Definition

$L_a(\varphi) = \{x \in (\{0,1\}^{m+n})^* \mid x \text{ is } m\text{-admissible and } \underline{x} \vDash \varphi\}$

Example

- $\varphi(x, X) = \forall y.\, y < x \to X(y)$
- $L_a(\varphi) = \binom{0}{1}^*[\binom{1}{0} + \binom{1}{1}][\binom{0}{0} + \binom{0}{1}]^*$

Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

Proof

induction on φ

- $\varphi = \bot \implies L_a(\varphi) = \varnothing$
- $\varphi = x < y \implies L_a(\varphi) = \binom{0}{0}^*\binom{1}{0}\binom{0}{0}^*\binom{0}{1}\binom{0}{0}^*$ or $L_a(\varphi) = \binom{0}{0}^*\binom{0}{1}\binom{0}{0}^*\binom{1}{0}\binom{0}{0}^*$
- $\varphi = X(x) \implies L_a(\varphi) = [\binom{0}{0} + \binom{0}{1}]^*\binom{1}{1}[\binom{0}{0} + \binom{0}{1}]^*$
- $\varphi = \neg\psi \implies L_a(\psi)$ is regular $\implies {\sim}L_a(\psi)$ is regular

  $\implies L_a(\varphi) = {\sim}L_a(\psi) \cap L(\mathcal{A}_{m,n})$ is regular (for suitable m and n)
- $\varphi = \varphi_1 \vee \varphi_2$ with $\mathrm{FV}(\varphi) = (x_1,\dots,x_m,X_1,\dots,X_n)$

  $L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

  applications of inverse homomorphism drop_i^{-1} to $L_a(\varphi_1)$ and $L_a(\varphi_2)$ yield regular sets $L_1, L_2 \subseteq (\{0,1\}^{m+n})^*$ such that $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{m,n})$

Example

$\varphi = x < y \vee X(x)$ with $\mathrm{FV}(\varphi) = (x, y, X)$

- $L_a(x < y) = \binom{0}{0}^* \binom{1}{0} \binom{0}{0}^* \binom{0}{1} \binom{0}{0}^*$
- $L_a(X(x)) = [\binom{0}{0} + \binom{0}{1}]^* \binom{1}{1} [\binom{0}{0} + \binom{0}{1}]^*$
- $L_1 = \mathrm{drop}_3^{-1}(\binom{0}{0}^* \binom{1}{0} \binom{0}{0}^* \binom{0}{1} \binom{0}{0}^*) = \begin{pmatrix}0\\0\\*\end{pmatrix}^* \begin{pmatrix}1\\0\\*\end{pmatrix} \begin{pmatrix}0\\0\\*\end{pmatrix}^* \begin{pmatrix}0\\1\\*\end{pmatrix} \begin{pmatrix}0\\0\\*\end{pmatrix}^*$
- $L_2 = \mathrm{drop}_2^{-1}([\binom{0}{0} + \binom{0}{1}]^* \binom{1}{1} [\binom{0}{0} + \binom{0}{1}]^*) = \left[\begin{pmatrix}0\\*\\0\end{pmatrix} + \begin{pmatrix}0\\*\\1\end{pmatrix}\right]^* \begin{pmatrix}1\\*\\1\end{pmatrix} \left[\begin{pmatrix}0\\*\\0\end{pmatrix} + \begin{pmatrix}0\\*\\1\end{pmatrix}\right]^*$
- $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{2,1})$

Definition

homomorphism

$$\mathrm{drop}_i \colon (\{0,1\}^k)^* \to (\{0,1\}^{k-1})^*$$

is defined for $1 \leqslant i \leqslant k$ by dropping i-th component from vectors in $\{0,1\}^k$

$$\mathrm{drop}_i \begin{pmatrix}a_1\\ \vdots\\ a_i\\ \vdots\\ a_k\end{pmatrix} = \begin{pmatrix}a_1\\ \vdots\\ \vdots\\ a_k\end{pmatrix}$$

Lemmata

- $A \subseteq (\{0,1\}^k)^*$ is regular $\implies$ $\mathrm{drop}_i(A) \subseteq (\{0,1\}^{k-1})^*$ is regular
- $B \subseteq (\{0,1\}^{k-1})^*$ is regular $\implies$ $\mathrm{drop}_i^{-1}(B) \subseteq (\{0,1\}^k)^*$ is regular

Proof (cont'd)

- $\varphi = \exists\, x.\, \psi$ $\implies$ $L_a(\psi)$ is regular
  $\implies$ $\mathrm{drop}_i(L_a(\psi))$ is regular where i is position of x in $\mathrm{FV}(\psi)$
  $\implies$ $L_a(\varphi) = \mathrm{stz}(\mathrm{drop}_i(L_a(\psi)))$ is regular
- $\varphi = \exists\, X.\, \psi$ $\implies$ $L_a(\psi)$ is regular
  $\implies$ $\mathrm{drop}_i(L_a(\psi))$ is regular where i is position of X in $\mathrm{FV}(\psi)$
  $\implies$ $L_a(\varphi) = \mathrm{stz}(\mathrm{drop}_i(L_a(\psi)))$ is regular

Example

$\varphi = \exists\, X.\, \psi$ with $\psi = X(x) \wedge \exists\, y.\, x < y \wedge X(y)$

- $L_a(\psi) = [\binom{0}{0} + \binom{0}{1}]^* \binom{1}{1} [\binom{0}{0} + \binom{0}{1}]^* \binom{0}{1} [\binom{0}{0} + \binom{0}{1}]^*$
- $\mathrm{drop}_2(L_a(\psi)) = (0+0)^* 1 (0+0)^* 0 (0+0)^* = 0^* 100^* \neq 0^* 10^* = L_a(\varphi)$

Definition

$\mathrm{stz}(A) = \{x \mid x\mathbf{0}\cdots\mathbf{0} \in A\}$ for $A \subseteq (\{0,1\}^{m+n})^*$ "shorten trailing zeros"

Lemma

$A \subseteq (\{0,1\}^k)^*$ is regular $\implies$ $\mathrm{stz}(A)$ is regular

Proof

- DFA $M = (Q, \Sigma, \delta, s, F)$ with $L(M) = A$
- construct DFA $M' = (Q, \Sigma, \delta, s, F')$ with $F' = \{q \in Q \mid \widehat{\delta}(q, x) \in F$ for some $x \in \mathbf{0}^*\}$
- $L(M') = \mathrm{stz}(A)$

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$ using regularity preserving operations

Procedure

① eliminate assignments which do not correspond to string in Σ^*

② map strings in 0^*10^* to elements of Σ using homomorphismm $h\colon \{0^k10^l \mid k+1+l = |\Sigma|\} \to \Sigma$ which maps 0^k10^l to k+1-th element of Σ

Lemma

$L(\varphi) = h(L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*)$ is regular

Corollary

WMSO definable sets are regular

Outline

MONA

- MONA is state-of-the-art tool that implements decision procedures for WS1S and WS2S
- WS1S is weak monadic second-order theory of 1 successor = WMSO
- MONA translates WS1S formulas into minimum-state DFAs
- MONA confirms validity or produces counterexample

Demo

`https://www.brics.dk/mona/`

Outline

Ebbinghaus, Flum and Thomas

- Section 10.9 of Einführung in die mathematische Logik (Springer Spektrum 2018)

Klarlund and Møller

- Section 3 of MONA Version 1.4 User Manual (2001)

Important Concepts

- drop_i
- $L_a(\varphi)$
- m-admissible
- MONA
- stz
- WS1S

homework for November 15