universität innsbruck

WS 2024 lecture 7

Automata and Logic

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Outline

1. **Summary of Previous Lecture**
2. **Presburger Arithmetic**
3. **Intermezzo**
4. **Presburger Arithmetic**
5. **WMSO**
6. **Further Reading**

Definitions

- $\mathrm{FV}(\varphi)$ denotes list of free variables in φ in fixed order with first-order variables preceding second-order ones
- assignment for φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is tuple $(i_1, \dots, i_m, I_1, \dots, I_n)$ such that $i_1, \dots, i_m$ are elements of $\mathbb{N}$ and $I_1, \dots, I_n$ are finite subsets of $\mathbb{N}$
- assignments are identified with strings over $\{0,1\}^{m+n}$
- string over $\{0,1\}^{m+n}$ is m-admissible if first m rows contain exactly one 1 each

Remarks

- every m-admissible string x induces assignment $\underline{x}$
- every assignment is induced by (not necessarily unique) m-admissible string: if x is m-admissible then $x\mathbf{0}$ is m-admissible and $\underline{x} = \underline{x\mathbf{0}}$

Lemma

set of m-admissible strings over $\{0,1\}^{m+n}$ is regular and accepted by DFA $\mathcal{A}_{m,n}$

Definition

$L_a(\varphi) = \{x \in (\{0,1\}^{m+n})^* \mid x \text{ is } m\text{-admissible and } \underline{x} \vDash \varphi\}$

Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

Definitions

- homomorphism $\mathrm{drop}_i\colon (\{0,1\}^k)^* \to (\{0,1\}^{k-1})^*$ is defined for $1 \leqslant i \leqslant k$ by dropping i-th component from vectors in $\{0,1\}^k$
- $\mathrm{stz}(A) = \{x \mid x\mathbf{0}\cdots\mathbf{0} \in A\} \supseteq A$ for $A \subseteq (\{0,1\}^{m+n})^*$ "shorten trailing zeros"

Lemma

$A \subseteq (\{0,1\}^k)^*$ is regular $\implies$ $\mathrm{stz}(A)$ is regular

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with $\mathrm{FV}(\varphi) = \{P_a \mid a \in \Sigma^*\}$ using regularity preserving operations

Procedure

① eliminate assignments which do not correspond to string in Σ^*

② map strings in 0^*10^* to elements of Σ using homorphism $h\colon \{0,1\}^{|\Sigma|} \to \Sigma$ which maps 0^k10^l to $k{+}1$-th element of Σ

Lemma

$L(\varphi) = h(L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*)$ is regular

Corollary

WMSO definable sets are regular

MONA

- MONA is state-of-the-art tool that implements decision procedures for WS1S and WS2S
- WS1S is weak monadic second-order theory of 1 successor = WMSO

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

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formulas of Presburger arithmetic

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Examples

1. $\exists\, y.\, x = y + y + y + 1$

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Abbreviations

$$\varphi \wedge \psi \;:=\; \neg(\neg\varphi \vee \neg\psi) \qquad \varphi \to \psi \;:=\; \neg\varphi \vee \psi \qquad \top \;:=\; \neg\bot$$

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$$\mathrm{n} := \underbrace{1 + \cdots + 1}_{n} \qquad \mathrm{n}\, x := \underbrace{x + \cdots + x}_{n} \quad \text{for } n > 1$$

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&\alpha \vDash \exists\, x.\, \varphi && \iff && \alpha[x \mapsto n] \vDash \varphi \text{ for some } n \in \mathbb{N} \\
&\alpha \vDash t_1 = t_2 && \iff && \alpha(t_1) = \alpha(t_2) \\
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Remark

$t_1 < t_2$ can be modeled as $\exists x.\, x \neq 0 \wedge t_1 + x = t_2$

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every $t_1 = t_2$ can be written as $a_1 x_1 + \cdots + a_n x_n = b$ with $a_1, \dots, a_n, b \in \mathbb{Z}$

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- induction

$$\psi(0) \wedge \forall x.\, (\psi(x) \rightarrow \psi(x+1)) \rightarrow \forall x.\, \psi(x)$$

  for every formula $\psi(x)$ with single free variable x

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  for every formula $\psi(x)$ with single free variable x
- $\forall x.\, x + 0 = x$
- $\forall x.\, \forall y.\, x + (y + 1) = (x + y) + 1$

Example

given natural numbers $a_1, \dots, a_n > 0$

$$(\forall y. x < y \to \exists x_1. \dots \exists x_n. a_1 x_1 + \cdots + a_n x_n = y) \land \neg(\exists x_1. \dots \exists x_n. a_1 x_1 + \cdots + a_n x_n = x)$$

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expresses largest number x that does not satisfy $a_1x_1 + \cdots + a_nx_n = x$ for some $x_1, \ldots, x_n \in \mathbb{N}$

Example (Frobenius Coin Problem)

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Theorem (Presburger 1929)

Presburger arithmetic is decidable

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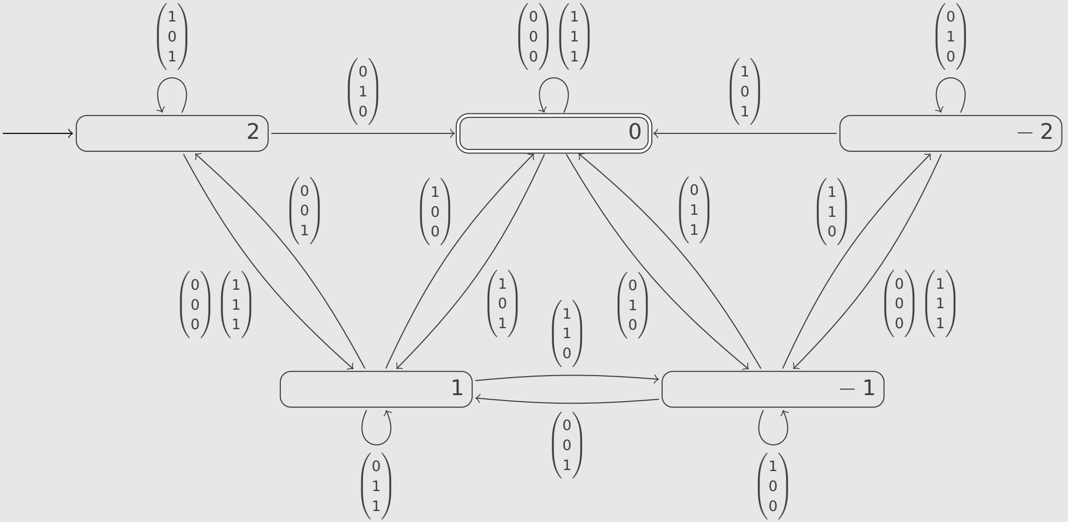
Example

Presburger arithmetic formula φ: $x + 2y - 3z = 2$

Example (cont'd)



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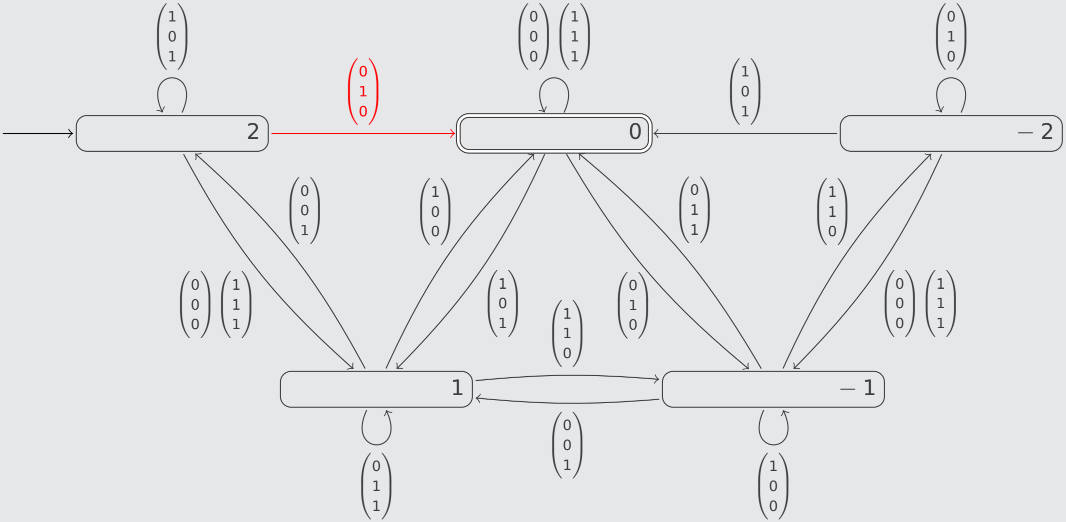

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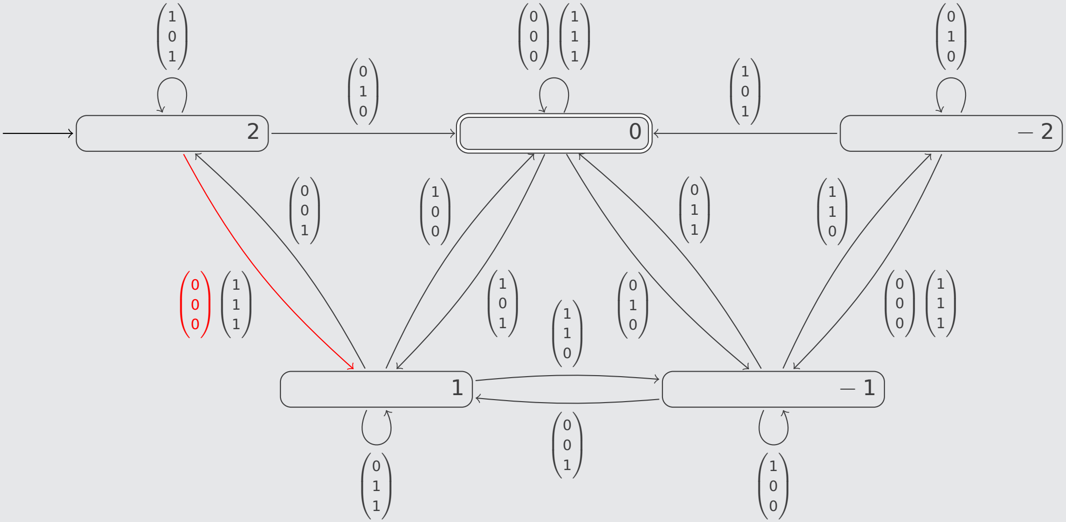

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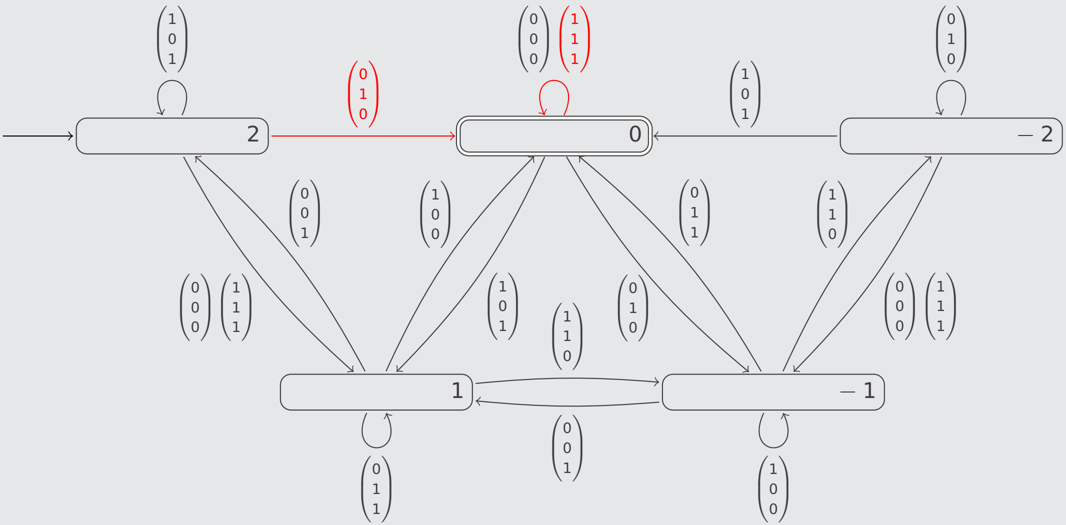

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- $\begin{pmatrix} \color{red}{0} \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \color{red}{1} \\ 1 \\ 1 \end{pmatrix}$ $x = (10)_2 = 2$

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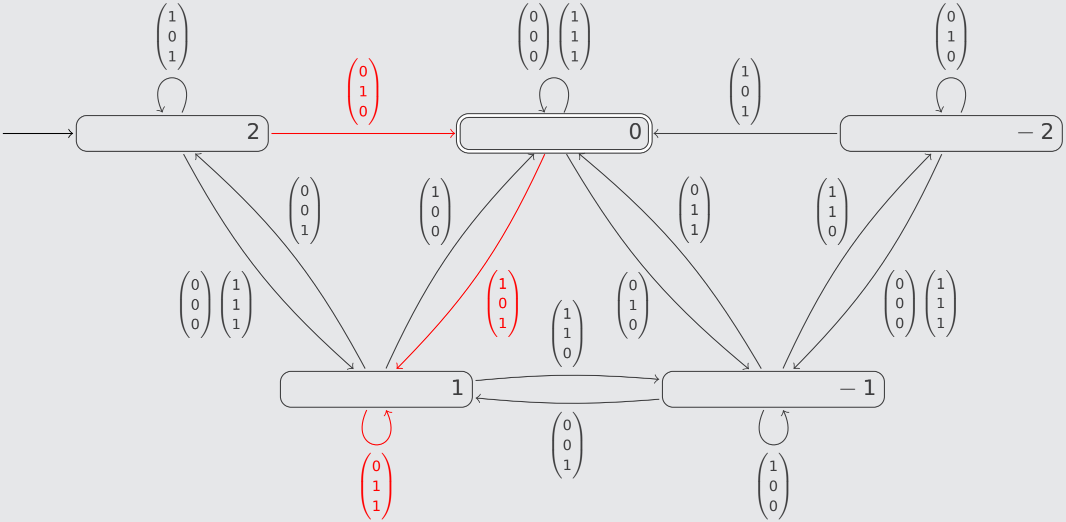

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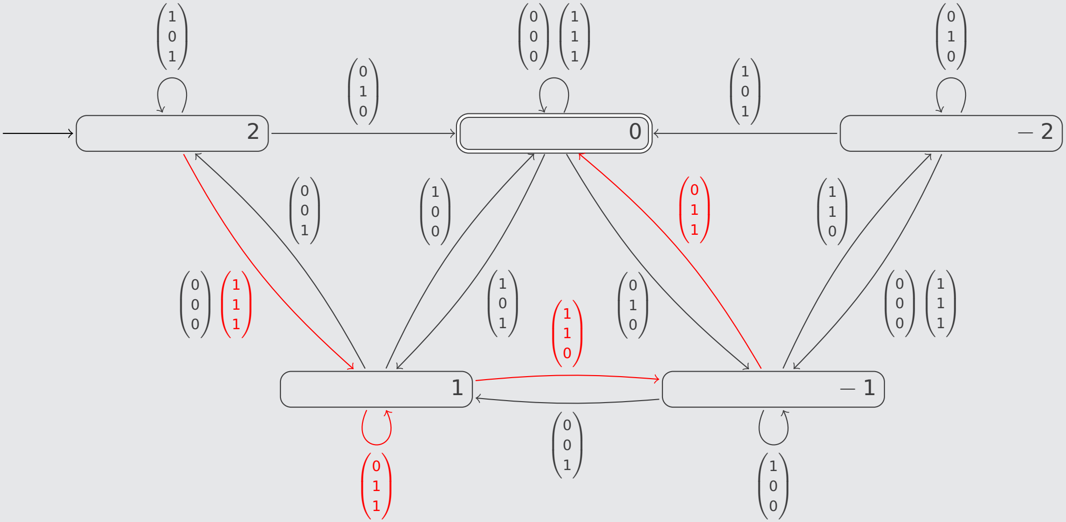

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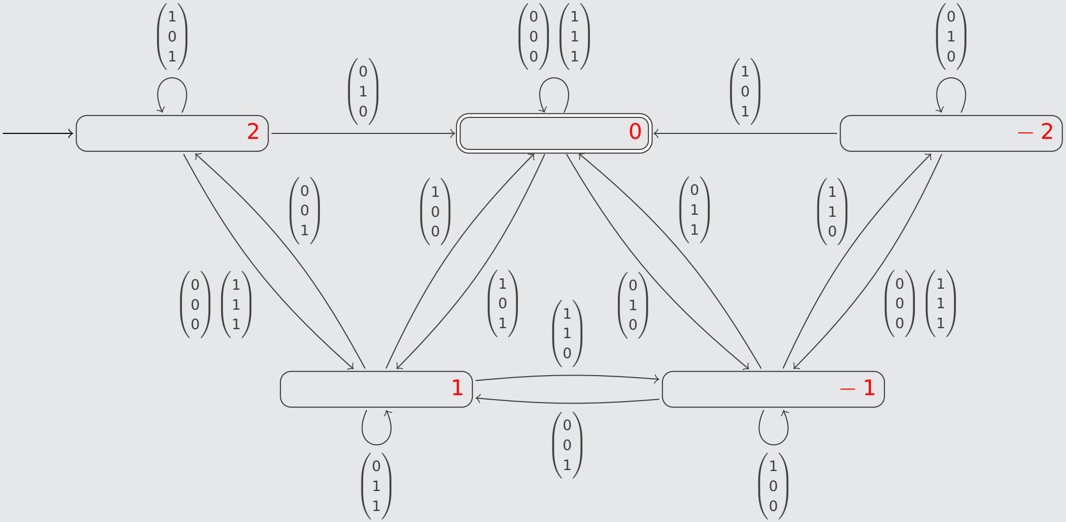

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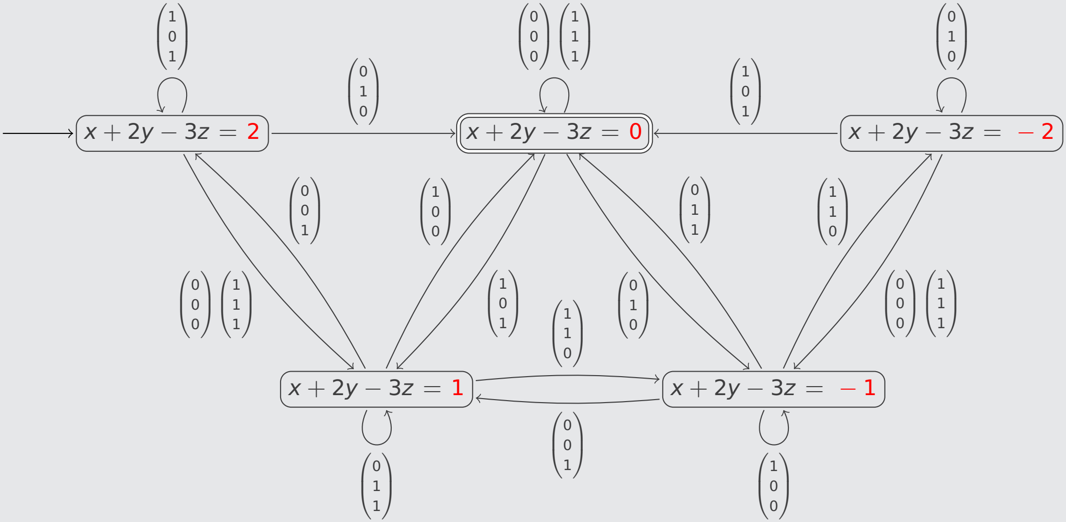

Definition (Representation)

- sequence of n natural numbers is represented as string over

$$\Sigma_{n} = \{(b_1 \cdots b_n)^{\mathsf{T}} \mid b_1, \ldots, b_n \in \{0, 1\}\}$$

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- $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ is represented by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Definition (Representation)

- sequence of n natural numbers is represented as string over

$$\Sigma_n = \{(b_1 \cdots b_n)^{\mathsf{T}} \mid b_1, \dots, b_n \in \{0,1\}\}$$

- $x = \begin{pmatrix} b_1^1 \\ \vdots \\ b_n^1 \end{pmatrix} \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix} \cdots \begin{pmatrix} b_1^m \\ \vdots \\ b_n^m \end{pmatrix} \in \Sigma_n^*$ represents $x_1 = (b_1^m \cdots b_1^2 b_1^1)_2, \ \dots, \ x_n = (b_n^m \cdots b_n^2 b_n^1)_2$
- $\underline{x} = (x_1, \dots, x_n)$

Example

- $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ represents $x_1 = 10$, $x_2 = 7$, $x_3 = 6$
- $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ is represented by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, ...

Definition

for Presburger arithmetic formula φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_n)$

$$L(\varphi) = \{x \in \Sigma_n^* \mid \underline{x} \vDash \varphi\}$$

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Theorem (Presburger 1929)

Presburger arithmetic is decidable

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- construct finite automaton A_φ for every Presburger arithmetic formula φ

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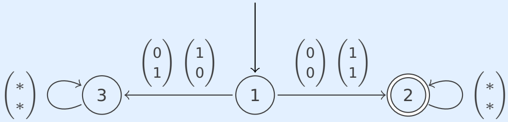
Outline

1. **Summary of Previous Lecture**
2. **Presburger Arithmetic**
3. **Intermezzo**
4. **Presburger Arithmetic**
5. **WMSO**
6. **Further Reading**

Question

Consider the following automaton A:



For which of the following formulas φ does $L(A) = L(\varphi)$ hold ?

A $x = y$

B $x + y > 0$

C $\exists z. x + y = 2z$

D $(\exists z. x = 2z \wedge \forall z. \neg(y = 2z)) \vee (\exists z. y = 2z \wedge \forall z. \neg(x = 2z))$

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   Atomic Formulas   Boolean Operations   Quantifiers

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Definition (Automaton for Atomic Formula)

DFA $A_\varphi = (Q, \Sigma_n, \delta, s, F)$ for $\varphi(x_1, \ldots, x_n)$: $a_1x_1 + \cdots + a_nx_n = b$

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if $\delta(i, (b_1 \cdots b_n)^\mathsf{T}) = j$ then $a_1 x_1 + \cdots + a_n x_n = j \iff a_1(2x_1 + b_1) + \cdots + a_n(2x_n + b_n) = i$

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Example

$\varphi(x, y)$: $x + 2y = 3$

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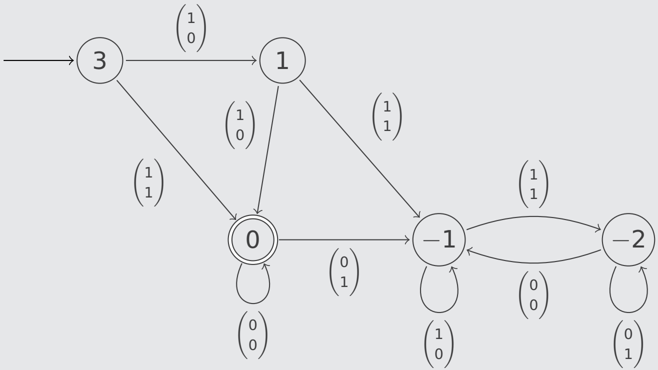

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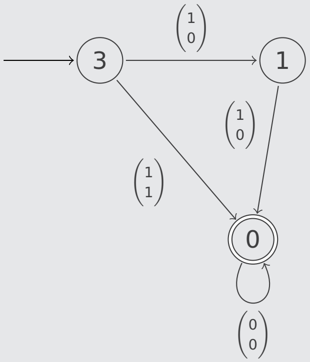

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$$|i - (a_1b_1 + \cdots + a_nb_n)| \leqslant |i| + |a_1b_1| + \cdots + |a_nb_n|$$

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Boolean Operations

boolean operation	automata construction
$\neg$	complement

Boolean Operations

boolean operation	automata construction
$\neg$	complement
$\wedge$	intersection

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boolean operation	automata construction
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Example

Presburger arithmetic formula

$$\neg((x + 2y - 3z \neq 2 \ \wedge \ 2x - y + z = 3) \ \vee \ x - 3y - z \neq 1))$$

Boolean Operations

boolean operation	automata construction	
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$\vee$	union	

Example

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Definition (Cylindrification)

$\mathrm{C}_i(R) \subseteq \Sigma_{n+1}^*$ is defined for $R \subseteq \Sigma_n^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\mathrm{C}_i(R) = \left\{ x_1 \cdots x_m \in \Sigma_{n+1}^* \;\middle|\; \mathrm{drop}_i(x_1) \cdots \mathrm{drop}_i(x_m) \in R \right\}$$

with $\mathrm{drop}_i\big((b_1 \cdots b_{n+1})^\mathsf{T}\big) = (b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1})^\mathsf{T}$

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Outline

1. **Summary of Previous Lecture**

2. **Presburger Arithmetic**

3. **Intermezzo**

4. **Presburger Arithmetic**

   Atomic Formulas   Boolean Operations   Quantifiers

5. **WMSO**

6. **Further Reading**

Definition (Projection)

$\Pi_i(R) \subseteq \Sigma_n^*$ is defined for $R \subseteq \Sigma_{n+1}^*$ and index $1 \leqslant i \leqslant n+1$ as

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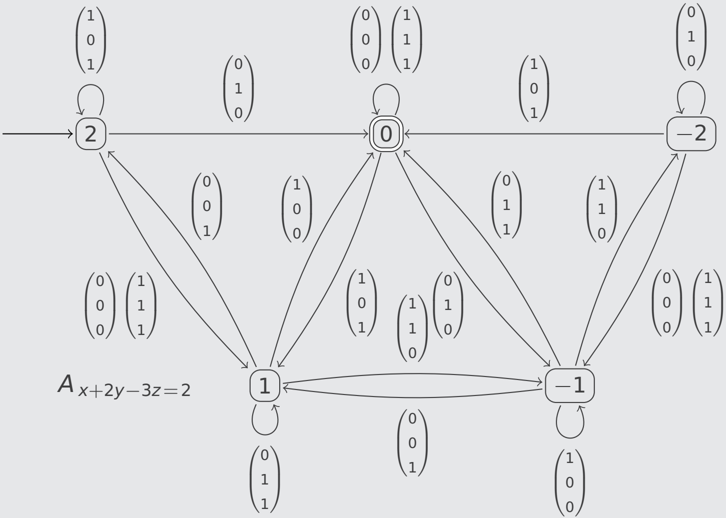

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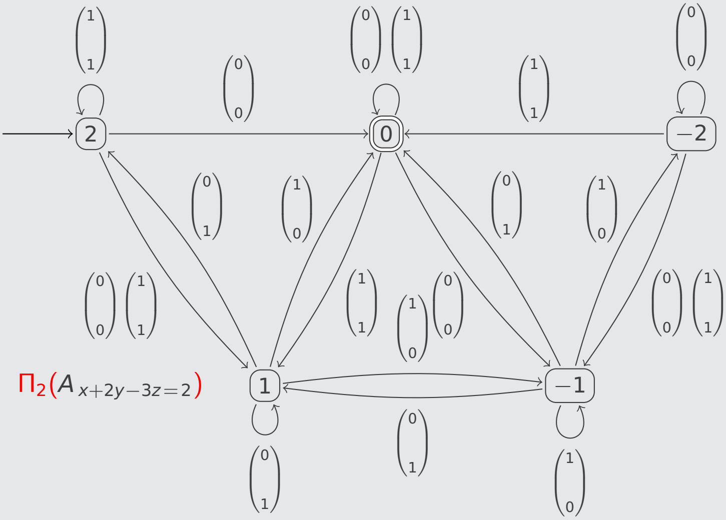

$\Pi_2(A_{x+2y-3z=2})$

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1. **Summary of Previous Lecture**

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6. **Further Reading**

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Decision Procedures

- quantifier elimination
- automata techniques
- translation to WMSO

Procedure

- map variables in Presburger arithmetic formula to second-order variables in WMSO

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- n is represented as set of "1" positions in reverse binary notation of n
- 0 and 1 in Presburger arithmetic formulas are translated into ZERO and ONE with

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- $+$ in Presburger arithmetic formula is translated into ternary predicate P_+ with

$$\begin{aligned} P_+(X,Y,Z) \;:=\; & \exists C.\, \neg C(0) \wedge \big(\forall x.\, C(x+1) \leftrightarrow X(x) \wedge Y(x) \vee X(x) \wedge C(x) \vee Y(x) \wedge C(x)\big) \wedge \\ & \big(\forall x.\, Z(x) \leftrightarrow X(x) \wedge Y(x) \wedge C(x) \vee X(x) \wedge \neg Y(x) \wedge \neg C(x) \vee \\ & \neg X(x) \wedge Y(x) \wedge \neg C(x) \vee \neg X(x) \wedge \neg Y(x) \wedge C(x)\big) \end{aligned}$$

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Example

Presburger arithmetic formula $\exists\, y.\, x = y + y + 1$ is transformed into WMSO formula

$$\big(\forall\, x.\, \mathrm{ONE}(x) \leftrightarrow x = 0\big) \;\wedge\; \exists\, Y.\, \exists\, Z.\, P_+(Y, Y, Z) \wedge P_+(Z, \mathrm{ONE}, X)$$

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Corollary

Presburger arithmetic is decidable

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Boudet and Comon

- Diophantine Equations, Presburger Arithmetic and Finite Automata, Proc. 21st International Colloquium on Trees in Algebra and Programming, LNCS 1059, pp. 30–43, 1996

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Important Concepts

- A_φ
- $L(\varphi)$
- cylindrification
- projection
- Presburger arithmetic

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homework for November 22