universität innsbruck

WS 2024 lecture 7

Automata and Logic

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Outline

Definitions

- $\mathsf{FV}(\varphi)$ denotes list of free variables in φ in fixed order with first-order variables preceding second-order ones
- assignment for φ with $\mathsf{FV}(\varphi) = (x_1,\dots,x_m,X_1,\dots,X_n)$ is tuple $(i_1,\dots,i_m,I_1,\dots,I_n)$ such that $i_1,\dots,i_m$ are elements of $\mathbb{N}$ and $I_1,\dots,I_n$ are finite subsets of $\mathbb{N}$
- assignments are identified with strings over $\{0,1\}^{m+n}$
- string over $\{0,1\}^{m+n}$ is m-admissible if first m rows contain exactly one 1 each

Remarks

- every m-admissible string x induces assignment $\underline{x}$
- every assignment is induced by (not necessarily unique) m-admissible string: if x is m-admissible then $x\mathbf{0}$ is m-admissible and $\underline{x} = \underline{x\mathbf{0}}$

Lemma

set of m-admissible strings over $\{0,1\}^{m+n}$ is regular and accepted by DFA $\mathcal{A}_{m,n}$

Definition

$L_a(\varphi) = \{x \in (\{0,1\}^{m+n})^* \mid x \text{ is } m\text{-admissible and } \underline{x} \vDash \varphi\}$

Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

Definitions

- homomorphism $\mathsf{drop}_i \colon (\{0,1\}^k)^* \to (\{0,1\}^{k-1})^*$ is defined for $1 \leqslant i \leqslant k$ by dropping i-th component from vectors in $\{0,1\}^k$
- $\mathsf{stz}(A) = \{x \mid x\mathbf{0}\cdots\mathbf{0} \in A\} \supseteq A$ for $A \subseteq (\{0,1\}^{m+n})^*$ "shorten trailing zeros"

Lemma

$A \subseteq (\{0,1\}^k)^*$ is regular $\implies$ stz(A) is regular

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with $\mathrm{FV}(\varphi) = \{P_a \mid a \in \Sigma^*\}$ using regularity preserving operations

Procedure

① eliminate assignments which do not correspond to string in Σ^*

② map strings in 0^*10^* to elements of Σ using homorphism $h\colon \{0,1\}^{|\Sigma|} \to \Sigma$ which maps 0^k10^l to $k{+}1$-th element of Σ

Lemma

$L(\varphi) = h(L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*)$ is regular

Corollary

WMSO definable sets are regular

MONA

- MONA is state-of-the-art tool that implements decision procedures for WS1S and WS2S
- WS1S is weak monadic second-order theory of 1 successor = WMSO

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Definition

formulas of Presburger arithmetic

$$\varphi ::= \bot \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists\, x.\, \varphi \mid t_1 = t_2 \mid t_1 < t_2$$
$$t ::= 0 \mid 1 \mid t_1 + t_2 \mid x$$

Examples

1. $\exists\, y.\, x = y + y + y + 1$
2. $\forall\, x.\, (\exists\, y.\, x = y + y) \vee (\exists\, y.\, x + 1 = y + y)$

Abbreviations

$$\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi) \qquad \varphi \to \psi := \neg\varphi \vee \psi \qquad \top := \neg\bot$$
$$\forall\, x.\, \varphi := \neg\exists\, x.\, \neg\varphi \qquad t_1 \leqslant t_2 := t_1 < t_2 \vee t_1 = t_2$$
$$n := \underbrace{1 + \cdots + 1}_{n} \qquad n\,x := \underbrace{x + \cdots + x}_{n} \quad \text{for } n > 1$$

Definitions

- assignment α is mapping from first-order variables to $\mathbb{N}$
- extension to terms: $\alpha(0) = 0 \quad \alpha(1) = 1 \quad \alpha(t_1 + t_2) = \alpha(t_1) + \alpha(t_2)$
- assignment α satisfies formula φ ($\alpha \vDash \varphi$):

$$\begin{aligned}
&\alpha \nvDash \bot \\
&\alpha \vDash \neg\varphi && \iff && \alpha \nvDash \varphi \\
&\alpha \vDash \varphi_1 \vee \varphi_2 && \iff && \alpha \vDash \varphi_1 \text{ or } \alpha \vDash \varphi_2 \\
&\alpha \vDash \exists\, x.\, \varphi && \iff && \alpha[x \mapsto n] \vDash \varphi \text{ for some } n \in \mathbb{N} \\
&\alpha \vDash t_1 = t_2 && \iff && \alpha(t_1) = \alpha(t_2) \\
&\alpha \vDash t_1 < t_2 && \iff && \alpha(t_1) < \alpha(t_2)
\end{aligned}$$

Remark

$t_1 < t_2$ can be modeled as $\exists\, x.\, x \neq 0 \wedge t_1 + x = t_2$

Remark

every $t_1 = t_2$ can be written as $a_1x_1 + \cdots + a_nx_n = b$ with $a_1, \ldots, a_n, b \in \mathbb{Z}$

Side Remark

Presburger arithmetic admits complete first-order axiomatization:

- $\forall\, x.\, x + 1 \neq 0$
- $\forall\, x.\, \forall\, y.\, x + 1 = y + 1 \to x = y$
- induction

$$\psi(0) \wedge \forall\, x.\, (\psi(x) \to \psi(x+1)) \to \forall\, x.\, \psi(x)$$

  for every formula $\psi(x)$ with single free variable x
- $\forall\, x.\, x + 0 = x$
- $\forall\, x.\, \forall\, y.\, x + (y + 1) = (x + y) + 1$

Example (Frobenius Coin Problem)

given natural numbers $a_1, \ldots, a_n > 0$

$$(\forall\, y.\, x < y \to \exists\, x_1.\, \ldots \exists\, x_n.\, a_1x_1 + \cdots + a_nx_n = y) \wedge \neg(\exists\, x_1.\, \ldots \exists\, x_n.\, a_1x_1 + \cdots + a_nx_n = x)$$

expresses largest number x that does not satisfy $a_1x_1 + \cdots + a_nx_n = x$ for some $x_1, \ldots, x_n \in \mathbb{N}$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Decision Procedures

- quantifier elimination
- automata techniques
- translation to WMSO

Example

Presburger arithmetic formula φ: $x + 2y - 3z = 2$

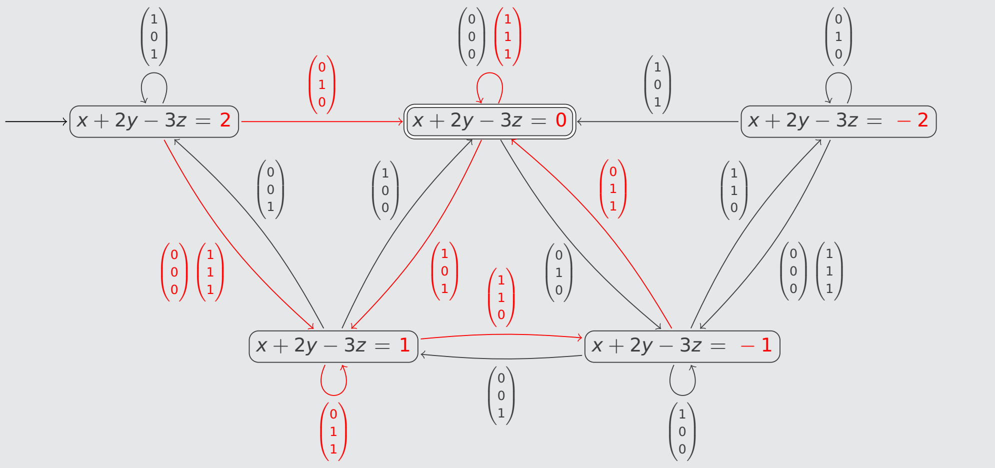
some accepted strings:

- $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $x = 0$ $y = 1$ $z = 0$
- $\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $x = (10)_2 = 2$ $y = (11)_2 = 3$ $z = (10)_2 = 2$
- $\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}$ $x = (0101)_2 = 5$ $y = (1111)_2 = 15$ $z = (1011)_2 = 11$

some rejected strings:

- $\begin{pmatrix}0\\0\\0\end{pmatrix}$ $x = 0$ $y = 0$ $z = 0$
- $\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}$ $x = (010)_2 = 2$ $y = (101)_2 = 5$ $z = (110)_2 = 6$

Example (cont'd)



Definition (Representation)

- sequence of n natural numbers is represented as string over

$$\Sigma_n = \{(b_1 \cdots b_n)^{\mathsf{T}} \mid b_1, \dots, b_n \in \{0,1\}\}$$

- $x = \begin{pmatrix}b_1^1\\ \vdots \\ b_n^1\end{pmatrix}\begin{pmatrix}b_1^2\\ \vdots \\ b_n^2\end{pmatrix}\cdots\begin{pmatrix}b_1^m\\ \vdots \\ b_n^m\end{pmatrix} \in \Sigma_n^*$ represents $x_1 = (b_1^m \cdots b_1^2 b_1^1)_2, \dots, x_n = (b_n^m \cdots b_n^2 b_n^1)_2$
- $\underline{x} = (x_1, \dots, x_n)$

Example

- $\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$ represents $x_1 = 10$, $x_2 = 7$, $x_3 = 6$
- $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ is represented by $\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}$, …

Definition

for Presburger arithmetic formula φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_n)$

$$L(\varphi) = \{\underline{x} \in \Sigma_n^* \mid \underline{x} \vDash \varphi\}$$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Proof Sketch

- construct finite automaton A_φ for every Presburger arithmetic formula φ
- induction on φ
- $L(A_\varphi) = L(\varphi)$

Outline

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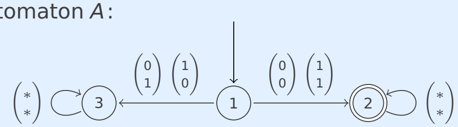
Question

Consider the following automaton A:



For which of the following formulas φ does $L(A) = L(\varphi)$ hold ?

A $x = y$

B $x + y > 0$

C $\exists z.\, x + y = 2z$

D $\big(\exists z.\, x = 2z \wedge \forall z.\, \neg(y = 2z)\big) \vee \big(\exists z.\, y = 2z \wedge \forall z.\, \neg(x = 2z)\big)$

Outline

Definition (Automaton for Atomic Formula)

DFA $A_\varphi = (Q, \Sigma_n, \delta, s, F)$ for $\varphi(x_1, \dots, x_n)$: $a_1 x_1 + \cdots + a_n x_n = b$

- $Q \subseteq \{\, i \mid |i| \leqslant |b| + |a_1| + \cdots + |a_n| \,\} \cup \{\bot\}$
- $\delta(i, (b_1 \cdots b_n)^{\mathsf{T}}) = \begin{cases} \dfrac{i - (a_1 b_1 + \cdots + a_n b_n)}{2} & \text{if } i - (a_1 b_1 + \cdots + a_n b_n) \text{ is even} \\ \bot & \text{if } i - (a_1 b_1 + \cdots + a_n b_n) \text{ is odd or } i = \bot \end{cases}$
- $s = b$
- $F = \{0\}$

Lemma

if $\delta(i, (b_1 \cdots b_n)^{\mathsf{T}}) = j$ then $a_1 x_1 + \cdots + a_n x_n = j \iff a_1(2x_1 + b_1) + \cdots + a_n(2x_n + b_n) = i$

Theorem

1. A_φ is well-defined
2. $L(A_\varphi) = L(\varphi)$

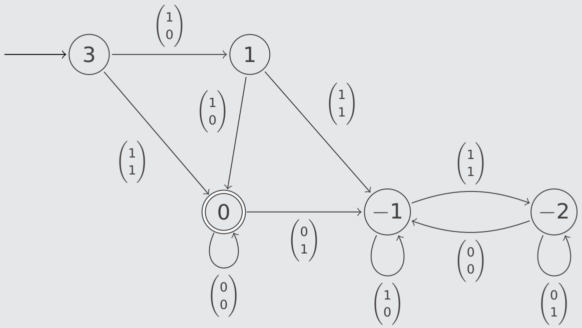
Example

$\varphi(x, y)$: $x + 2y = 3$

- $Q = \{3, \bot, 1, 0, -1, -2\}$ $\quad s = 3 \quad F = \{0\}$
- $\delta(3, \binom{0}{0}) = \delta(3, \binom{0}{1}) = \bot \quad \delta(3, \binom{1}{0}) = 1 \quad \delta(3, \binom{1}{1}) = 0$
- $\delta(\bot, \binom{0}{0}) = \delta(\bot, \binom{0}{1}) = \delta(\bot, \binom{1}{0}) = \delta(\bot, \binom{1}{1}) = \bot$
- $\delta(1, \binom{0}{0}) = \delta(1, \binom{0}{1}) = \bot \quad \delta(1, \binom{1}{0}) = 0 \quad \delta(1, \binom{1}{1}) = -1$
- $\delta(0, \binom{1}{0}) = \delta(0, \binom{1}{1}) = \bot \quad \delta(0, \binom{0}{0}) = 0 \quad \delta(0, \binom{0}{1}) = -1$
- $\delta(-1, \binom{0}{0}) = \delta(-1, \binom{0}{1}) = \bot \quad \delta(-1, \binom{1}{0}) = -1 \quad \delta(-1, \binom{1}{1}) = -2$
- $\delta(-2, \binom{1}{0}) = \delta(-2, \binom{1}{1}) = \bot \quad \delta(-2, \binom{0}{0}) = -1 \quad \delta(-2, \binom{0}{1}) = -2$

Example

$\varphi(x, y)$: $x + 2y = 3$



Theorem

1. A_φ is well-defined
2. $L(A_\varphi) = L(\varphi)$

Proof

- $q \in Q \subseteq \{\, i \mid |i| \leqslant |b| + |a_1| + \cdots + |a_n| \,\} \cup \{\bot\}$ and $b' = (b_1 \cdots b_n)^{\mathsf{T}} \in \Sigma_n$
- if $q = \bot$ or $i - (a_1 b_1 + \cdots + a_n b_n)$ is odd then $\delta(q, b') = \bot$
- suppose $q = i$ with $|i| \leqslant |b| + |a_1| + \cdots + |a_n|$ and $i - (a_1 b_1 + \cdots + a_n b_n)$ is even

$$\delta(i, b') = \frac{i - (a_1 b_1 + \cdots + a_n b_n)}{2} \in Q$$

$$\begin{aligned} |i - (a_1 b_1 + \cdots + a_n b_n)| &\leqslant |i| + |a_1 b_1| + \cdots + |a_n b_n| \\ &\leqslant |i| + |a_1| + \cdots + |a_n| \\ &\leqslant 2\,(|b| + |a_1| + \cdots + |a_n|) \end{aligned}$$

Outline

Boolean Operations

boolean operation	automata construction	
$\neg$	complement	C
$\wedge$	intersection	I
$\vee$	union	U

Example

Presburger arithmetic formula

$$\neg((x + 2y - 3z \neq 2 \;\wedge\; 2x - y + z = 3) \;\vee\; x - 3y - z \neq 1))$$

is implemented as

$$\mathsf{C}(\mathsf{U}(\mathsf{I}(\mathsf{C}(A_{x+2y-3z=2}), A_{2x-y+z=3}), \mathsf{C}(A_{x-3y-z=1})))$$

Example

Presburger arithmetic formula

$$x + 2y - 3z = 2 \;\wedge\; x + 2y = 3$$

- $A_{x+2y-3z=2}$ operates on alphabet $\Sigma_3 = (\{0,1\}^3)^\mathsf{T}$
- $A_{x+2y=3}$ operates on alphabet $\Sigma_2 = (\{0,1\}^2)^\mathsf{T}$
- before intersection can be computed $A_{x+2y=3}$ needs to operate on Σ_3

Definition (Cylindrification)

$\mathsf{C}_i(R) \subseteq \Sigma_{n+1}^*$ is defined for $R \subseteq \Sigma_n^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\mathsf{C}_i(R) = \left\{ x_1 \cdots x_m \in \Sigma_{n+1}^* \;\middle|\; \mathsf{drop}_i(x_1) \cdots \mathsf{drop}_i(x_m) \in R \right\}$$

with $\mathsf{drop}_i((b_1 \cdots b_{n+1})^\mathsf{T}) = (b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1})^\mathsf{T}$

Example

- $L(A_{x+2y=3}) = \left\{ \binom{1}{1}, \binom{1}{0}\binom{1}{0} \right\} \left\{ \binom{0}{0} \right\}^*$
- $L(\mathsf{C}_3(A_{x+2y=3})) = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} \right\} \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}^*$

Lemma

if $R \subseteq \Sigma_n^*$ is regular then $\mathsf{C}_i(R) \subseteq \Sigma_{n+1}^*$ is regular for every $1 \leqslant i \leqslant n+1$

Remark

- drop_i is homomorphism from Σ_{n+1}^* to Σ_n^*
- $\mathsf{C}_i(R) = \mathsf{drop}_i^{-1}(R)$

Outline

Definition (Projection)

$\Pi_i(R) \subseteq \Sigma_n^*$ is defined for $R \subseteq \Sigma_{n+1}^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\Pi_i(R) = \{\mathrm{drop}_i(x_1)\cdots \mathrm{drop}_i(x_m) \in \Sigma_n^* \mid x_1 \cdots x_m \in R\}$$

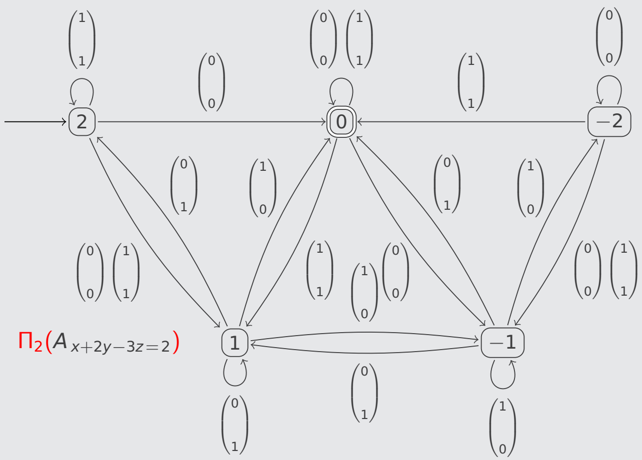
Lemma

if $R \subseteq \Sigma_{n+1}^*$ is regular then $\Pi_i(R) \subseteq \Sigma_n^*$ is regular for every $1 \leqslant i \leqslant n+1$

Example

1. solutions of $\exists\, y.\, x + 2y - 3z = 2$ correspond to $\mathrm{stz}(\Pi_2(A_{x+2y-3z=2}))$
2. solutions of $\forall\, y.\, x + 2y - 3z = 2$ correspond to $\mathrm{C}(\mathrm{stz}(\Pi_2(\mathrm{C}(A_{x+2y-3z=2}))))$

Example



Outline

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Decision Procedures

- quantifier elimination
- automata techniques
- translation to WMSO

Procedure

- map variables in Presburger arithmetic formula to second-order variables in WMSO
- n is represented as set of "1" positions in reverse binary notation of n
- 0 and 1 in Presburger arithmetic formulas are translated into ZERO and ONE with

$$\forall x.\, \neg \mathrm{ZERO}(x) \qquad\qquad \forall x.\, \mathrm{ONE}(x) \leftrightarrow x = 0$$

- $+$ in Presburger arithmetic formula is translated into ternary predicate P_+ with

$$\begin{aligned} P_+(X,Y,Z) :=\ & \exists C.\, \neg C(0) \wedge \big(\forall x.\, C(x+1) \leftrightarrow X(x) \wedge Y(x) \vee X(x) \wedge C(x) \vee Y(x) \wedge C(x)\big) \wedge \\ & \big(\forall x.\, Z(x) \leftrightarrow X(x) \wedge Y(x) \wedge C(x) \vee X(x) \wedge \neg Y(x) \wedge \neg C(x) \vee \\ & \neg X(x) \wedge Y(x) \wedge \neg C(x) \vee \neg X(x) \wedge \neg Y(x) \wedge C(x)\big) \end{aligned}$$

Example

26 is represented by $\{1,3,4\}$ since $(26)_2 = 11010$

Example

Presburger arithmetic formula $\exists y.\, x = y + y + 1$ is transformed into WMSO formula

$$\big(\forall x.\, \mathrm{ONE}(x) \leftrightarrow x = 0\big) \wedge \exists Y.\, \exists Z.\, P_+(Y,Y,Z) \wedge P_+(Z,\mathrm{ONE},X)$$

Corollary

Presburger arithmetic is decidable

Outline

Boudet and Comon

- Diophantine Equations, Presburger Arithmetic and Finite Automata, Proc. 21st International Colloquium on Trees in Algebra and Programming, LNCS 1059, pp. 30–43, 1996

Esparza and Blondin

- Chapter 9 of Automata Theory: An Algorithmic Approach (MIT Press 2023)

Important Concepts

- A_φ
- $L(\varphi)$
- cylindrification
- projection
- Presburger arithmetic

homework for November 22