WS 2024 lecture 8

Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

1. **Summary of Previous Lecture**
2. **Infinite Strings**
3. **Büchi Automata**
4. **Intermezzo**
5. **Closure Properties**
6. **Further Reading**

Definition

formulas of Presburger arithmetic

$$\varphi ::= \bot \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists\, x.\, \varphi \mid t_1 = t_2 \mid t_1 < t_2$$
$$t ::= 0 \mid 1 \mid t_1 + t_2 \mid x$$

Abbreviations

$$\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi) \qquad \varphi \rightarrow \psi := \neg\varphi \vee \psi \qquad \top := \neg\bot$$
$$\forall\, x.\, \varphi := \neg\exists\, x.\, \neg\varphi \qquad t_1 \leqslant t_2 := t_1 < t_2 \vee t_1 = t_2$$
$$\mathrm{n} := \underbrace{1 + \cdots + 1}_{n} \qquad \mathrm{n}\, x := \underbrace{x + \cdots + x}_{n} \qquad \text{for } n > 1$$

Definitions

- assignment α is mapping from first-order variables to $\mathbb{N}$
- extension to terms: $\alpha(0) = 0 \qquad \alpha(1) = 1 \qquad \alpha(t_1 + t_2) = \alpha(t_1) + \alpha(t_2)$

Definition

assignment α satisfies formula φ ($\alpha \vDash \varphi$):

$$
\begin{aligned}
&\alpha \nvDash \bot \\
&\alpha \vDash \neg\varphi && \iff && \alpha \nvDash \varphi \\
&\alpha \vDash \varphi_1 \vee \varphi_2 && \iff && \alpha \vDash \varphi_1 \text{ or } \alpha \vDash \varphi_2 \\
&\alpha \vDash \exists x.\, \varphi && \iff && \alpha[x \mapsto n] \vDash \varphi \text{ for some } n \in \mathbb{N} \\
&\alpha \vDash t_1 = t_2 && \iff && \alpha(t_1) = \alpha(t_2) \\
&\alpha \vDash t_1 < t_2 && \iff && \alpha(t_1) < \alpha(t_2)
\end{aligned}
$$

Remark

every $t_1 = t_2$ can be written as $a_1x_1 + \cdots + a_nx_n = b$ with $a_1, \ldots, a_n, b \in \mathbb{Z}$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Decision Procedures

- quantifier elimination
- automata techniques
- translation to WMSO

Definition (Representation)

- sequence of n natural numbers is represented as string over

$$\Sigma_n = \{(b_1 \cdots b_n)^{\mathsf{T}} \mid b_1, \dots, b_n \in \{0, 1\}\}$$

- $x = \begin{pmatrix} b_1^1 \\ \vdots \\ b_n^1 \end{pmatrix} \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix} \cdots \begin{pmatrix} b_1^m \\ \vdots \\ b_n^m \end{pmatrix} \in \Sigma_n^*$ represents $x_1 = (b_1^m \cdots b_1^2 b_1^1)_2, \dots, x_n = (b_n^m \cdots b_n^2 b_n^1)_2$
- $\underline{x} = (x_1, \dots, x_n)$

Definition

for Presburger arithmetic formula φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_n)$

$$L(\varphi) = \{x \in \Sigma_n^* \mid \underline{x} \vDash \varphi\}$$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Proof Sketch

- construct finite automaton A_φ for every Presburger arithmetic formula φ
- induction on φ
- $L(A_\varphi) = L(\varphi)$

Definition (Automaton for Atomic Formula)

finite automaton $A_\varphi = (Q, \Sigma_n, \delta, s, F)$ for $\varphi(x_1, \dots, x_n)$: $a_1x_1 + \cdots + a_nx_n = b$

- $Q \subseteq \{i \mid |i| \leqslant |b| + |a_1| + \cdots + |a_n|\} \cup \{\bot\}$
- $\delta(i, (b_1 \cdots b_n)^\mathsf{T}) = \begin{cases} \dfrac{i - (a_1b_1 + \cdots + a_nb_n)}{2} & \text{if } i - (a_1b_1 + \cdots + a_nb_n) \text{ is even} \\ \bot & \text{if } i - (a_1b_1 + \cdots + a_nb_n) \text{ is odd or } i = \bot \end{cases}$
- $s = b$
- $F = \{0\}$

Lemma

if $\delta(i, (b_1 \cdots b_n)^\mathsf{T}) = j$ then $a_1x_1 + \cdots + a_nx_n = j \iff a_1(2x_1 + b_1) + \cdots + a_n(2x_n + b_n) = i$

Theorem

- A_φ is well-defined
- $L(A_\varphi) = L(\varphi)$

Boolean Operations

boolean operation	automata construction	
$\neg$	complement	C
$\wedge$	intersection	I
$\vee$	union	U

Definition (Cylindrification)

$\mathsf{C}_i(R) \subseteq \Sigma_{n+1}^*$ is defined for $R \subseteq \Sigma_n^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\mathsf{C}_i(R) = \left\{ x_1 \cdots x_m \in \Sigma_{n+1}^* \;\middle|\; \mathrm{drop}_i(x_1) \cdots \mathrm{drop}_i(x_m) \in R \right\}$$

with $\mathrm{drop}_i\big((b_1 \cdots b_{n+1})^{\mathsf{T}}\big) = (b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1})^{\mathsf{T}}$

Lemma

if $R \subseteq \Sigma_n^*$ is regular then $\mathsf{C}_i(R) \subseteq \Sigma_{n+1}^*$ is regular for every $1 \leqslant i \leqslant n+1$

Definition (Projection)

$\Pi_i(R) \subseteq \Sigma_n^*$ is defined for $R \subseteq \Sigma_{n+1}^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\Pi_i(R) = \{\mathrm{drop}_i(x_1) \cdots \mathrm{drop}_i(x_m) \in \Sigma_n^* \mid x_1 \cdots x_m \in R\}$$

Lemma

if $R \subseteq \Sigma_{n+1}^*$ is regular then $\Pi_i(R) \subseteq \Sigma_n^*$ is regular for every $1 \leqslant i \leqslant n+1$

Translation from Presburger Arithmetic to WMSO

- map variables in Presburger arithmetic formula to second-order variables in WMSO
- n is represented as set of "1" positions in reverse binary notation of n
- 0 and 1 in Presburger arithmetic formulas are translated into ZERO and ONE with

$$\forall x. \neg \mathrm{ZERO}(x) \qquad \forall x. \mathrm{ONE}(x) \leftrightarrow x = 0$$

- $+$ in Presburger arithmetic formula is translated into ternary predicate P_+ with

$$\begin{aligned} P_+(X,Y,Z) := \exists\, C. \neg C(0) \wedge \big(\forall x. C(x+1) \leftrightarrow X(x) \wedge Y(x) \vee X(x) \wedge C(x) \vee Y(x) \wedge C(x)\big) \wedge \\ \big(\forall x. Z(x) \leftrightarrow X(x) \wedge Y(x) \wedge C(x) \vee X(x) \wedge \neg Y(x) \wedge \neg C(x) \vee \\ \neg X(x) \wedge Y(x) \wedge \neg C(x) \vee \neg X(x) \wedge \neg Y(x) \wedge C(x)\big) \end{aligned}$$

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

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- infinite string x is identified with infinite sequence $x(0)x(1)x(2)\cdots$

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- $|x|_a$ for $x \in \Sigma^\omega$ and $a \in \Sigma$ denotes number of occurrences of a in x

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- $|x|_a = \infty$ for at least one $a \in \Sigma$

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- $\sim V = \Sigma^\omega - V$ is complement of $V \subseteq \Sigma^\omega$
- $U^\omega = \{u_0 \cdot u_1 \cdot \cdots \mid u_i \in U - \{\epsilon\} \text{ for all } i \in \mathbb{N}\}$ is ω-iteration of $U \subseteq \Sigma^*$

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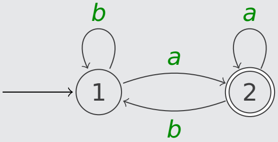
- NBA M



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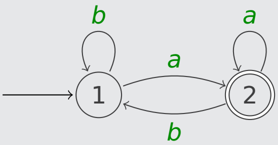

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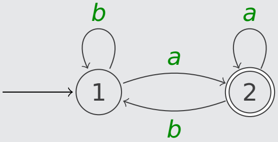

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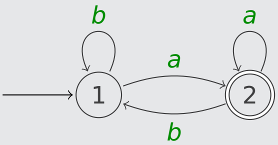

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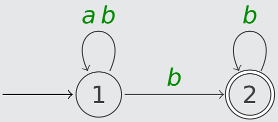

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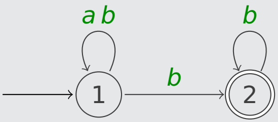

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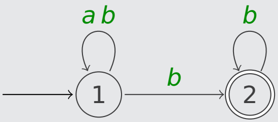

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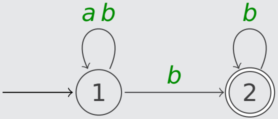

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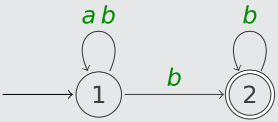

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- deterministic Büchi automaton (DBA) is NBA $(Q, \Sigma, \Delta, S, F)$ with

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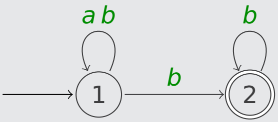

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- M is not deterministic

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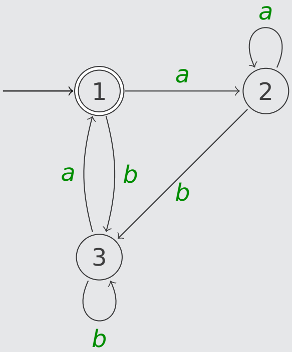
$\{x \in \{a, b\}^\omega \mid |x|_a = |x|_b = \infty\}$ is accepted by DBA



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let $l_1 = i_1 - i_0 - 1$

$x_2 = b^{i_0} a b^{l_1} a b^\omega$

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$x_1 = b^{i_0} a b^\omega \in L \implies \exists$ accepting run $q_0, q_1, \dots \implies \exists\, i_1 > i_0 + 1$ with $q_{i_1} \in F$

let $l_1 = i_1 - i_0 - 1$

$x_2 = b^{i_0} a b^{l_1} a b^\omega \in L \implies \exists$ accepting run $q_0, q_1, \cdots \implies \exists\, i_2 > i_1 + 1$ with $q_{i_2} \in F$

Theorem

not every ω-regular set is accepted by DBA

Proof

$L = \{x \in \{a, b\}^\omega \mid |x|_a \neq \infty\}$ is ω-regular but not accepted by DBA:

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$\exists\, j < k$ such that $q_{i_j} = q_{i_k}$

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- $x = b^{i_0} a b^{l_1} \cdots a b^{l_j} (a b^{l_j + 1} \cdots a b^{l_k})^\omega$

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  $x_1 = b^{i_0}ab^\omega \in L \implies \exists$ accepting run $q_0, q_1, \dots \implies \exists\, i_1 > i_0 + 1$ with $q_{i_1} \in F$

  let $l_1 = i_1 - i_0 - 1$

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Lemma

every ω-regular set is accepted by NBA with one start state

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Proof

- $A = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$
- define NBA $N = (Q', \Sigma, \Delta', \{s\}, F)$ with $Q' = Q \uplus \{s\}$ and

$$\Delta'(p,a) = \begin{cases} \Delta(p,a) & \text{if } p \neq s \\ \{q \in Q \mid q \in \Delta(p',a) \text{ for some } p' \in S\} & \text{if } p = s \end{cases}$$

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$$x \in A \quad \Longleftrightarrow \quad \exists \text{ run } q_0, q_1, q_2, \ldots \text{ in } M \text{ with } q_0 \in S \text{ and } q_i \in F \text{ for infinitely many } i \geqslant 0$$

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Proof

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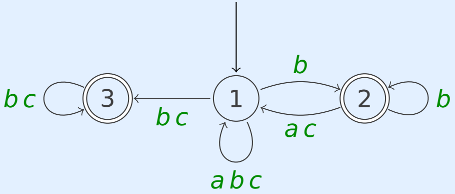
Outline

Particify with session ID **8020 8256**

Question

Which statement about the following NBA M is true ?



A $L(M) = \varnothing$

B $L(M) = \{x \mid |x|_b = \infty \text{ and } |x|_c = \infty\}$

C $L(M) = \{x \mid |x|_a \neq \infty \text{ or } |x|_a = |x|_b = \infty\}$

D $L(M) = \Sigma^\omega$

Outline

Theorem

ω-regular sets are effectively closed under union

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ω-regular sets are effectively closed under union

Proof (construction)

- $A = L(M_1)$ for NBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

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- $A \cup B = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$ with

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  ④ $\Delta(q, a) = \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \\ \Delta_2(q, a) & \text{if } q \in Q_2 \end{cases}$

Theorem

ω-regular sets are effectively closed under intersection

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Remark

product construction needs to be modified

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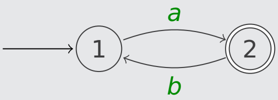
product construction needs to be modified

Example

M_1 :



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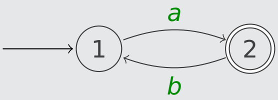

$$L(M_1) = a(ba)^\omega = (ab)^\omega$$

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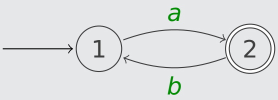
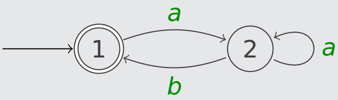


M_2:



$L(M_1) = a(ba)^\omega = (ab)^\omega$ $\qquad$ $L(M_2) = (aa^*b)^\omega$

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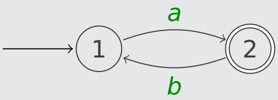
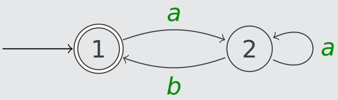


M_2 :



$L(M_1) = a(ba)^\omega = (ab)^\omega \qquad L(M_2) = (aa^*b)^\omega \qquad L(M_1) \cap L(M_2) = (ab)^\omega$

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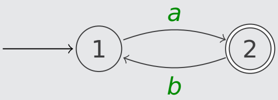
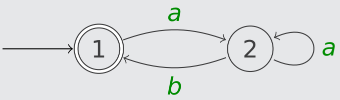
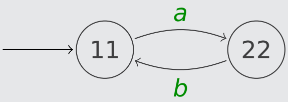


M_2:



$L(M_1) = a(ba)^\omega = (ab)^\omega$ $\quad$ $L(M_2) = (aa^*b)^\omega$ $\quad$ $L(M_1) \cap L(M_2) = (ab)^\omega$

$M_1 \times M_2$:



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Proof (modified product construction)

- $A = L(M_1)$ for NBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$

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  3. $F = Q_1 \times Q_2 \times \{2\}$
  4. $\Delta((p, q, i), a) = \{(p', q', j) \mid p' \in \Delta_1(p, a)$ and $q' \in \Delta_2(q, a)\}$

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  3. $F = Q_1 \times Q_2 \times \{2\}$
  4. $\Delta((p, q, i), a) = \{(p', q', j) \mid p' \in \Delta_1(p, a) \text{ and } q' \in \Delta_2(q, a)\}$ with

$$j = \begin{cases} 1 & \text{if } i = 0 \text{ and } p' \in F_1 \text{ or } i = 1 \text{ and } q' \notin F_2 \\ & \\ & \end{cases}$$

Theorem

ω-regular sets are effectively closed under intersection

Proof (modified product construction)

- $A = L(M_1)$ for NBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- $A \cap B = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$ with
  1. $Q = Q_1 \times Q_2 \times \{0, 1, 2\}$
  2. $S = S_1 \times S_2 \times \{0\}$
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$$j = \begin{cases} 1 & \text{if } i = 0 \text{ and } p' \in F_1 \text{ or } i = 1 \text{ and } q' \notin F_2 \\ 2 & \text{if } i = 1 \text{ and } q' \in F_2 \end{cases}$$

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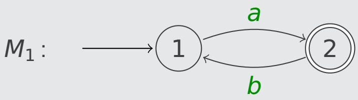
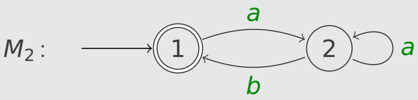
Example





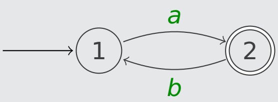
$L(M_1) = a(ba)^\omega = (ab)^\omega$

$L(M_2) = (aa^*b)^\omega$

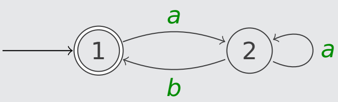
$L(M_1) \cap L(M_2) = (ab)^\omega$

Example

$M_1:$



$M_2:$



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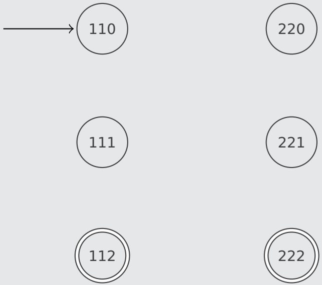

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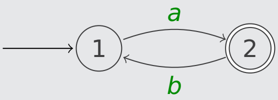
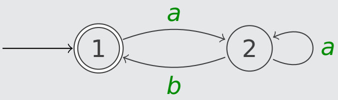


$M_2:$



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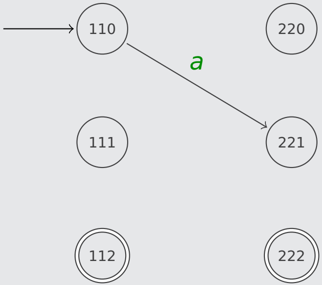

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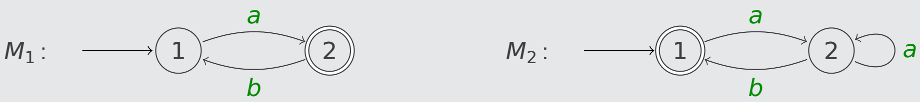

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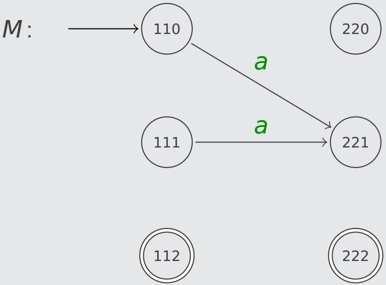

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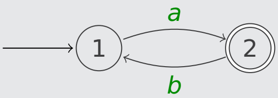
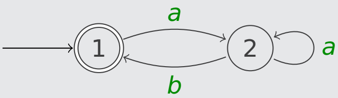


$M_2 :$



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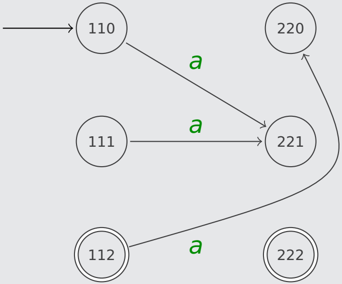

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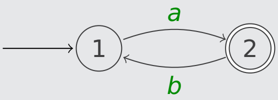
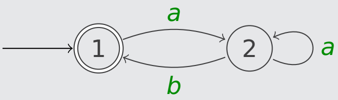


$M_2:$



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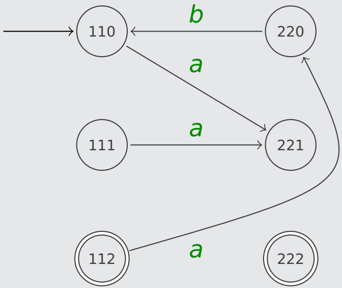

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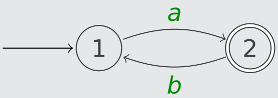

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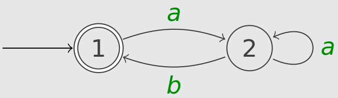

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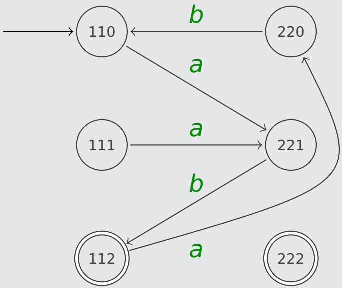

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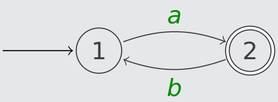

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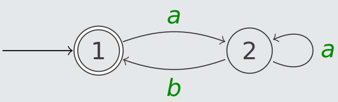

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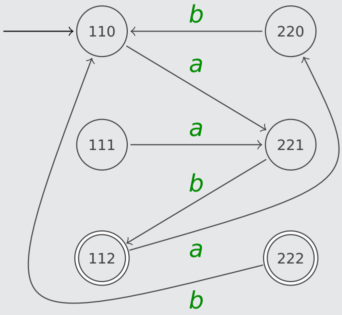

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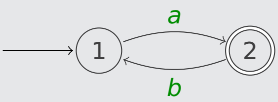

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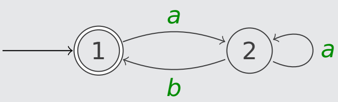

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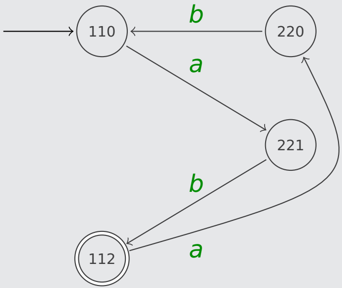

Theorem

left-concatenation of regular set and ω-regular set is ω-regular

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Proof (construction)

- $A = L(M_1)$ for NFA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$

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  ④ $\Delta = \Delta_1 \cup \Delta_2 \cup \{(p,a,q) \mid (p,a,f) \in \Delta_1 \text{ for some } f \in F_1 \text{ and } q \in S_2\}$

Theorem

ω-iteration of regular set is ω-regular

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Proof (construction)

- $A = L(M)$ for NFA $M = (Q, \Sigma, \Delta, S, F)$

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- $A = L(M)$ for NFA $M = (Q, \Sigma, \Delta, S, F)$
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Theorem

ω-iteration of regular set is ω-regular

Proof (construction)

- $A = L(M)$ for NFA $M = (Q, \Sigma, \Delta, S, F)$
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- NFA $M' = (Q \cup \{s\}, \Sigma, \Delta', \{s\}, F)$ with

$$\Delta' = \Delta \cup \{(s, a, q) \mid (p, a, q) \in \Delta \text{ for some } p \in S\}$$

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- $L(M') = L(M)$

Theorem

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- $L(M'') = L(M')^\omega$

Theorem

set $A \subseteq \Sigma^\omega$ is ω-regular $\iff$ $A = U_1 \cdot V_1^\omega \cup \cdots \cup U_n \cdot V_n^\omega$ for some $n \geqslant 0$ and regular $U_1, \dots, U_n, V_1, \dots, V_n \subseteq \Sigma^*$

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Proof ($\Longleftarrow$)

A is ω-regular using closure properties

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Proof ($\Longleftarrow$)

A is ω-regular using closure properties: ω-iteration

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Proof ($\Longleftarrow$)

A is ω-regular using closure properties: ω-iteration, left-concatenation

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Proof ($\Longleftarrow$)

A is ω-regular using closure properties: ω-iteration, left-concatenation, union

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- L_{pq} is regular for all $p, q \in Q$
- $A = \bigcup_{p \in S,\, q \in F} L_{pq} \cdot L_{qq}^\omega$

Outline

Hofmann and Lange

- Chapter 5 of Automatentheorie und Logik (Springer, 2011)

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Esparza and Blondin

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Important Concepts

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- DBA
- left-concatenation
- NBA
- Σ^ω
- ω-iteration
- ω-regular

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homework for November 29