Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

Definition

formulas of Presburger arithmetic

$$\varphi \ ::= \ \bot \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists\, x.\, \varphi \mid t_1 = t_2 \mid t_1 < t_2$$

$$t \ ::= \ 0 \mid 1 \mid t_1 + t_2 \mid x$$

Abbreviations

$$\varphi \wedge \psi \ := \ \neg(\neg\varphi \vee \neg\psi) \qquad \varphi \rightarrow \psi \ := \ \neg\varphi \vee \psi \qquad \top \ := \ \neg\bot$$

$$\forall\, x.\, \varphi \ := \ \neg\exists\, x.\, \neg\varphi \qquad t_1 \leqslant t_2 \ := \ t_1 < t_2 \vee t_1 = t_2$$

$$\mathrm{n} \ := \ \underbrace{1 + \cdots + 1}_{n} \qquad \mathrm{n}\, x \ := \ \underbrace{x + \cdots + x}_{n} \qquad \text{for } n > 1$$

Definitions

- assignment α is mapping from first-order variables to $\mathbb{N}$
- extension to terms: $\alpha(0) = 0 \qquad \alpha(1) = 1 \qquad \alpha(t_1 + t_2) = \alpha(t_1) + \alpha(t_2)$

Definition

assignment α satisfies formula φ ($\alpha \vDash \varphi$):

$$
\begin{aligned}
&\alpha \nvDash \bot \\
&\alpha \vDash \neg\varphi && \Longleftrightarrow && \alpha \nvDash \varphi \\
&\alpha \vDash \varphi_1 \vee \varphi_2 && \Longleftrightarrow && \alpha \vDash \varphi_1 \text{ or } \alpha \vDash \varphi_2 \\
&\alpha \vDash \exists x.\, \varphi && \Longleftrightarrow && \alpha[x \mapsto n] \vDash \varphi \text{ for some } n \in \mathbb{N} \\
&\alpha \vDash t_1 = t_2 && \Longleftrightarrow && \alpha(t_1) = \alpha(t_2) \\
&\alpha \vDash t_1 < t_2 && \Longleftrightarrow && \alpha(t_1) < \alpha(t_2)
\end{aligned}
$$

Remark

every $t_1 = t_2$ can be written as $a_1x_1 + \cdots + a_nx_n = b$ with $a_1, \ldots, a_n, b \in \mathbb{Z}$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Decision Procedures

- quantifier elimination
- automata techniques
- translation to WMSO

Definition (Representation)

- sequence of n natural numbers is represented as string over

$$\Sigma_n = \{(b_1 \cdots b_n)^{\mathsf{T}} \mid b_1, \ldots, b_n \in \{0,1\}\}$$

- $x = \begin{pmatrix} b_1^1 \\ \vdots \\ b_n^1 \end{pmatrix} \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix} \cdots \begin{pmatrix} b_1^m \\ \vdots \\ b_n^m \end{pmatrix} \in \Sigma_n^*$ represents $x_1 = (b_1^m \cdots b_1^2 b_1^1)_2, \ \ldots, \ x_n = (b_n^m \cdots b_n^2 b_n^1)_2$
- $\underline{x} = (x_1, \ldots, x_n)$

Definition

for Presburger arithmetic formula φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_n)$

$$L(\varphi) = \{x \in \Sigma_n^* \mid \underline{x} \vDash \varphi\}$$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Proof Sketch

- construct finite automaton A_φ for every Presburger arithmetic formula φ
- induction on φ
- $L(A_\varphi) = L(\varphi)$

Definition (Automaton for Atomic Formula)

finite automaton $A_\varphi = (Q, \Sigma_n, \delta, s, F)$ for $\varphi(x_1, \dots, x_n)$: $a_1x_1 + \cdots + a_nx_n = b$

- $Q \subseteq \{i \mid |i| \leqslant |b| + |a_1| + \cdots + |a_n|\} \cup \{\bot\}$
- $\delta(i, (b_1 \cdots b_n)^\mathsf{T}) = \begin{cases} \dfrac{i - (a_1b_1 + \cdots + a_nb_n)}{2} & \text{if } i - (a_1b_1 + \cdots + a_nb_n) \text{ is even} \\ \bot & \text{if } i - (a_1b_1 + \cdots + a_nb_n) \text{ is odd or } i = \bot \end{cases}$
- $s = b$
- $F = \{0\}$

Lemma

if $\delta(i, (b_1 \cdots b_n)^\mathsf{T}) = j$ then $a_1x_1 + \cdots + a_nx_n = j \iff a_1(2x_1 + b_1) + \cdots + a_n(2x_n + b_n) = i$

Theorem

- A_φ is well-defined
- $L(A_\varphi) = L(\varphi)$

Boolean Operations

boolean operation	automata construction	
$\neg$	complement	C
$\wedge$	intersection	I
$\vee$	union	U

Definition (Cylindrification)

$\mathsf{C}_i(R) \subseteq \Sigma_{n+1}^*$ is defined for $R \subseteq \Sigma_n^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\mathsf{C}_i(R) = \left\{ x_1 \cdots x_m \in \Sigma_{n+1}^* \;\middle|\; \mathsf{drop}_i(x_1) \cdots \mathsf{drop}_i(x_m) \in R \right\}$$

with $\mathsf{drop}_i\big((b_1 \cdots b_{n+1})^\mathsf{T}\big) = (b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1})^\mathsf{T}$

Lemma

if $R \subseteq \Sigma_n^*$ is regular then $\mathsf{C}_i(R) \subseteq \Sigma_{n+1}^*$ is regular for every $1 \leqslant i \leqslant n+1$

Definition (Projection)

$\Pi_i(R) \subseteq \Sigma_n^*$ is defined for $R \subseteq \Sigma_{n+1}^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\Pi_i(R) = \{\mathrm{drop}_i(x_1)\cdots \mathrm{drop}_i(x_m) \in \Sigma_n^* \mid x_1 \cdots x_m \in R\}$$

Lemma

if $R \subseteq \Sigma_{n+1}^*$ is regular then $\Pi_i(R) \subseteq \Sigma_n^*$ is regular for every $1 \leqslant i \leqslant n+1$

Translation from Presburger Arithmetic to WMSO

- map variables in Presburger arithmetic formula to second-order variables in WMSO
- n is represented as set of "1" positions in reverse binary notation of n
- 0 and 1 in Presburger arithmetic formulas are translated into ZERO and ONE with

$$\forall x.\, \neg \mathrm{ZERO}(x) \qquad\qquad \forall x.\, \mathrm{ONE}(x) \leftrightarrow x = 0$$

- $+$ in Presburger arithmetic formula is translated into ternary predicate P_+ with

$$\begin{aligned} P_+(X,Y,Z) \; := \exists\, C.\, \neg C(0) \wedge \big(\forall\, x.\, C(x+1) \leftrightarrow X(x) \wedge Y(x) \vee X(x) \wedge C(x) \vee Y(x) \wedge C(x)\big) \wedge \\ \big(\forall\, x.\, Z(x) \leftrightarrow X(x) \wedge Y(x) \wedge C(x) \vee X(x) \wedge \neg Y(x) \wedge \neg C(x) \vee \\ \neg X(x) \wedge Y(x) \wedge \neg C(x) \vee \neg X(x) \wedge \neg Y(x) \wedge C(x)\big) \end{aligned}$$

Automata

- (deterministic, non-deterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second-order logic
- Presburger arithmetic
- linear-time temporal logic

Outline

Definitions

- infinite string over alphabet Σ is function $x\colon \mathbb{N} \to \Sigma$
- Σ^ω denotes set of all infinite strings over Σ
- $|x|_a$ for $x \in \Sigma^\omega$ and $a \in \Sigma$ denotes number of occurrences of a in x

Example

$$x(i) = \begin{cases} a & \text{if } i \text{ is even} \\ b & \text{if } i \text{ is odd} \end{cases} \qquad x = ababab\cdots = (ab)^\omega$$

Remarks

- infinite string x is identified with infinite sequence $x(0)x(1)x(2)\cdots$
- $|x|_a = \infty$ for at least one $a \in \Sigma$

Definitions

- left-concatenation of $u \in \Sigma^*$ and $v \in \Sigma^\omega$ is denoted by $u \cdot v \in \Sigma^\omega$
- left-concatenation of $U \subseteq \Sigma^*$ and $V \subseteq \Sigma^\omega$

$$U \cdot V = \{u \cdot v \mid u \in U \text{ and } v \in V\}$$

- $\sim V = \Sigma^\omega - V$ is complement of $V \subseteq \Sigma^\omega$
- $U^\omega = \{u_0 \cdot u_1 \cdot \cdots \mid u_i \in U - \{\epsilon\} \text{ for all } i \in \mathbb{N}\}$ is ω-iteration of $U \subseteq \Sigma^*$

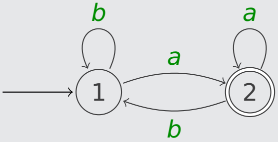
Outline

Definitions

- nondeterministic Büchi automaton (NBA) is NFA $M = (Q, \Sigma, \Delta, S, F)$ operating on Σ^ω
- run of M on input $x = a_0 a_1 a_2 \cdots \in \Sigma^\omega$ is infinite sequence $q_0, q_1, \dots$ of states such that $q_0 \in S$ and $q_{i+1} \in \Delta(q_i, a_i)$ for $i \geqslant 0$
- run $q_0, q_1, \dots$ is accepting if $q_i \in F$ for infinitely many i
- $L(M) = \{x \in \Sigma^\omega \mid x \text{ admits accepting run}\}$

Example

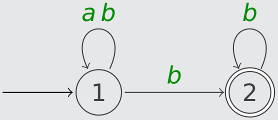
- NBA M



- $(ab)^\omega \in L(M)$
- $aab^\omega \notin L(M)$
- $L(M) = \{x \in \{a, b\}^\omega \mid |x|_a = \infty\}$

Example

- NBA M



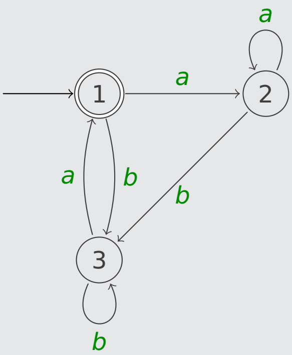
- $L(M) = \{x \in \{a,b\}^\omega \mid |x|_a \neq \infty\} = (a+b)^* b^\omega$
- M is not deterministic

Definitions

- set $A \subseteq \Sigma^\omega$ is ω-regular if $A = L(M)$ for some NBA M
- deterministic Büchi automaton (DBA) is NBA $(Q, \Sigma, \Delta, S, F)$ with
  1. $|S| = 1$
  2. $|\Delta(q,a)| = 1$ for all $q \in Q$ and $a \in \Sigma$

Example

$\{x \in \{a, b\}^\omega \mid |x|_a = |x|_b = \infty\}$ is accepted by DBA



Theorem

not every ω-regular set is accepted by DBA

Proof

$L = \{x \in \{a,b\}^\omega \mid |x|_a \neq \infty\}$ is ω-regular but not accepted by DBA:

- suppose $L = L(M)$ for DBA $M = (Q, \Sigma, \Delta, S, F)$

  $x_0 = b^\omega \in L \implies \exists$ accepting run $q_0, q_1, \ldots \implies \exists\, i_0 \geqslant 0$ with $q_{i_0} \in F$

  $x_1 = b^{i_0} a b^\omega \in L \implies \exists$ accepting run $q_0, q_1, \ldots \implies \exists\, i_1 > i_0 + 1$ with $q_{i_1} \in F$

  let $l_1 = i_1 - i_0 - 1$

  $x_2 = b^{i_0} a b^{l_1} a b^\omega \in L \implies \exists$ accepting run $q_0, q_1, \cdots \implies \exists\, i_2 > i_1 + 1$ with $q_{i_2} \in F$

  ...

  $\exists\, j < k$ such that $q_{i_j} = q_{i_k}$

- $x = b^{i_0} a b^{l_1} \cdots a b^{l_j} (a b^{l_j+1} \cdots a b^{l_k})^\omega$ admits accepting run but $x \notin L$ ↯

Lemma

every ω-regular set is accepted by NBA with one start state

Proof

- $A = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$
- define NBA $N = (Q', \Sigma, \Delta', \{s\}, F)$ with $Q' = Q \uplus \{s\}$ and

$$\Delta'(p,a) = \begin{cases} \Delta(p,a) & \text{if } p \neq s \\ \{q \in Q \mid q \in \Delta(p',a) \text{ for some } p' \in S\} & \text{if } p = s \end{cases}$$

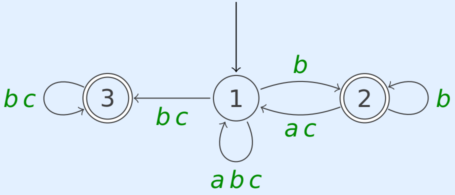
- $L(N) = A$:

$$\begin{aligned} x \in A \quad & \iff \quad \exists \text{ run } q_0, q_1, q_2, \dots \text{ in } M \text{ with } q_0 \in S \text{ and } q_i \in F \text{ for infinitely many } i \geqslant 0 \\ & \iff \quad \exists \text{ run } q_0, q_1, q_2, \dots \text{ in } M \text{ with } q_0 \in S \text{ and } q_i \in F \text{ for infinitely many } i > 0 \\ & \iff \quad \exists \text{ run } s, q_1, q_2, \dots \text{ in } N \text{ with } q_i \in F \text{ for infinitely many } i > 0 \\ & \iff \quad x \in L(N) \end{aligned}$$

Outline

Question

Which statement about the following NBA M is true ?



A $L(M) = \varnothing$

B $L(M) = \{x \mid |x|_b = \infty \text{ and } |x|_c = \infty\}$

C $L(M) = \{x \mid |x|_a \neq \infty \text{ or } |x|_a = |x|_b = \infty\}$

D $L(M) = \Sigma^\omega$

Outline

Theorem

ω-regular sets are effectively closed under union

Proof (construction)

- $A = L(M_1)$ for NBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

  $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- without loss of generality $Q_1 \cap Q_2 = \varnothing$
- $A \cup B = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$ with

  ① $Q = Q_1 \cup Q_2$

  ② $S = S_1 \cup S_2$

  ③ $F = F_1 \cup F_2$

  ④ $\Delta(q, a) = \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \\ \Delta_2(q, a) & \text{if } q \in Q_2 \end{cases}$

Theorem

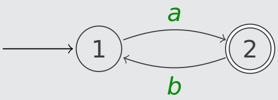
ω-regular sets are effectively closed under intersection

Remark

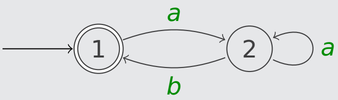
product construction needs to be modified

Example

M_1:

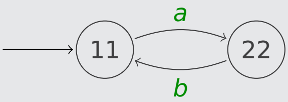


M_2:



$L(M_1) = a(ba)^\omega = (ab)^\omega$ $\quad$ $L(M_2) = (aa^*b)^\omega$ $\quad$ $L(M_1) \cap L(M_2) = (ab)^\omega$

$M_1 \times M_2$:



Theorem

ω-regular sets are effectively closed under intersection

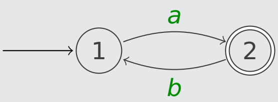
Proof (modified product construction)

- $A = L(M_1)$ for NBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- $A \cap B = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$ with
  1. $Q = Q_1 \times Q_2 \times \{0, 1, 2\}$
  2. $S = S_1 \times S_2 \times \{0\}$
  3. $F = Q_1 \times Q_2 \times \{2\}$
  4. $\Delta((p, q, i), a) = \{(p', q', j) \mid p' \in \Delta_1(p, a) \text{ and } q' \in \Delta_2(q, a)\}$ with

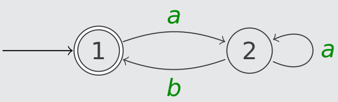
$$j = \begin{cases} 1 & \text{if } i = 0 \text{ and } p' \in F_1 \text{ or } i = 1 \text{ and } q' \notin F_2 \\ 2 & \text{if } i = 1 \text{ and } q' \in F_2 \\ 0 & \text{otherwise} \end{cases}$$

Example

M_1 :

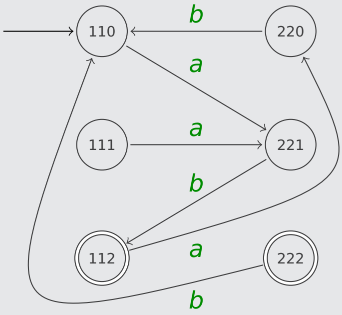


M_2 :



$L(M_1) = a(ba)^\omega = (ab)^\omega$ $\quad$ $L(M_2) = (aa^*b)^\omega$ $\quad$ $L(M_1) \cap L(M_2) = (ab)^\omega$

M :



Theorem

left-concatenation of regular set and ω-regular set is ω-regular

Proof (construction)

- $A = L(M_1)$ for NFA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- without loss of generality $Q_1 \cap Q_2 = \varnothing$
- $A \cdot B = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$ with

① $Q = Q_1 \cup Q_2$

② $S = \begin{cases} S_1 & \text{if } F_1 \cap S_1 = \varnothing \\ S_1 \cup S_2 & \text{otherwise} \end{cases}$

③ $F = F_2$

④ $\Delta = \Delta_1 \cup \Delta_2 \cup \{(p,a,q) \mid (p,a,f) \in \Delta_1 \text{ for some } f \in F_1 \text{ and } q \in S_2\}$

Theorem

ω-iteration of regular set is ω-regular

Proof (construction)

- $A = L(M)$ for NFA $M = (Q, \Sigma, \Delta, S, F)$
- without loss of generality $\epsilon \notin A$
- NFA $M' = (Q \cup \{s\}, \Sigma, \Delta', \{s\}, F)$ with

$$\Delta' = \Delta \cup \{(s, a, q) \mid (p, a, q) \in \Delta \text{ for some } p \in S\}$$

- $L(M') = L(M)$
- NBA $M'' = (Q \cup \{s\}, \Sigma, \Delta'', \{s\}, \{s\})$ with

$$\Delta'' = \Delta' \cup \{(p, a, s) \mid (p, a, q) \in \Delta' \text{ for some } q \in F\}$$

- $L(M'') = L(M')^{\omega}$

Theorem

set $A \subseteq \Sigma^\omega$ is ω-regular $\iff$ $A = U_1 \cdot V_1^\omega \cup \cdots \cup U_n \cdot V_n^\omega$ for some $n \geqslant 0$ and regular $U_1, \dots, U_n, V_1, \dots, V_n \subseteq \Sigma^*$

Proof ($\Longleftarrow$)

A is ω-regular using closure properties: ω-iteration, left-concatenation, union

Proof ($\Longrightarrow$)

- $A = L(M)$ for some NBA $M = (Q, \Sigma, \Delta, S, F)$
- L_{pq} for $p, q \in Q$ is set of strings $x \in \Sigma^*$ such that $q \in \widehat{\Delta}(\{p\}, x)$
- L_{pq} is regular for all $p, q \in Q$
- $A = \bigcup_{p \in S,\, q \in F} L_{pq} \cdot L_{qq}^\omega$

Outline

Hofmann and Lange

- Chapter 5 of Automatentheorie und Logik (Springer, 2011)

Esparza and Blondin

- Chapter 10 of Automata Theory: An Algorithmic Approach (MIT Press 2023)

Important Concepts

- Büchi automaton
- DBA
- left-concatenation
- NBA
- Σ^ω
- ω-iteration
- ω-regular

homework for November 29