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# Functional Programming

## Week 3 – Functions on Trees

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## Last Lecture

- data = tree shaped data
- every value, expression, function has a type
- types of `lhs` and `rhs` have to be equal in function definition `lhs = rhs`
- built-in types: `Int`, `Integer`, `Float`, `Double`, `String`, `Char`, `Bool`
- user defined datatypes

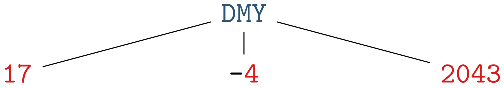
```
data TName =
    CName1 type1_1 ... type1_N1
  | ...
  | CNameM typeM_1 ... typeM_NM
  deriving Show
```

- constructor `CNameI :: typeI_1 -> ... -> typeI_NI -> TName` is a function that is not evaluated
- `TName` is recursive if some `typeI_J` is `TName`
- names of types and constructors start with uppercase letters

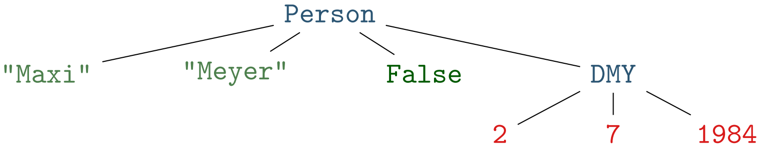
# Examples of Nonrecursive Datatype Definitions

```
data Date = DMY Int Int Integer deriving Show
data Person = Person String String Bool Date deriving Show
```

- values of type `Date` are trees such as



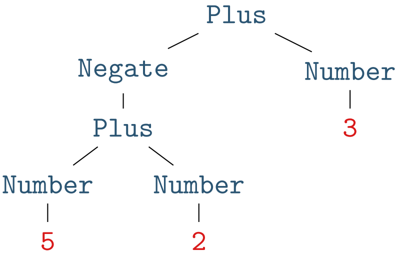
- values of type `Person` are trees such as



# Example of Recursive Datatype Definition – `Expr`

```
data Expr =
    Number Integer
  | Plus Expr Expr
  | Negate Expr
  deriving Show
```

- represent mathematical expression $(-(5+2))+3$ in Haskell (as value of type `Expr`):
  `Plus (Negate (Plus (Number 5) (Number 2))) (Number 3)`
- same expression as tree

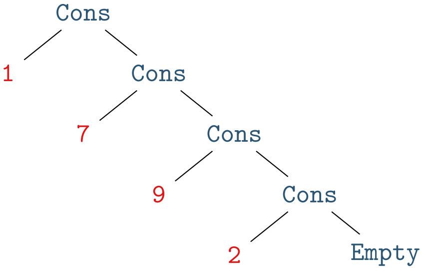


# Example of Recursive Datatype Definition – Lists

- lists are just a special kind of trees, e.g., lists of `Integer`s

```
data List =
    Empty
  | Cons Integer List
  deriving Show
```

- example representation of list $[1, 7, 9, 2]$
  - in Haskell: `Cons 1 (Cons 7 (Cons 9 (Cons 2 Empty)))`
  - as tree:



# Function Definitions Revisited

# Function Definitions and Expressions

- so far all functions definitions have been of the shape
  `funName x1 ... xN = expr`
  where
  - `x1` ... `xN` are variable names;
    a function can have arbitrary many parameters (including zero)
  - `expr` is an expression, i.e., a mathematical expression consisting of
    - variables: `x`, `y`, `xs`, `f`, ...
    - literals: `5`, `3.4`, `'a'`, `"hello"`, ...
    - function applications: `pi`, `square expr`, `average expr1 expr2`, ...
    - constructor applications: `True`, `Number expr`, `Cons expr1 expr2`, ...
    - operator applications: `- expr`, `expr1 + expr2`, ...
    - parenthesis
  - remark: function and constructor applications bind stronger than operator applications

    `(square 2) + 4` = `square 2 + 4` ≠ `square (2 + 4)`
- this lecture: extend shape of function definitions,
  in particular to define functions on tree shaped data

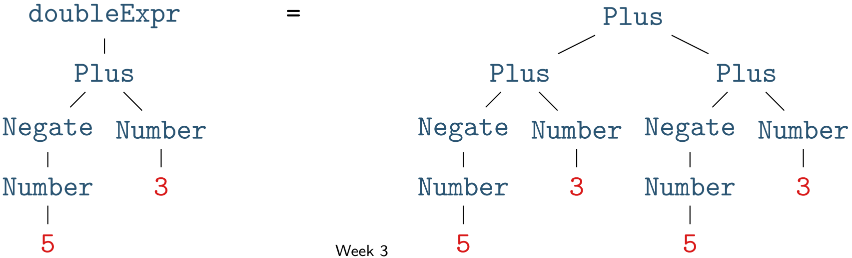
# Creating New Values – `Expr` Example

- creation of new values is easily possible using constructors
- example: consider `Expr` datatype

  ```
  data Expr = Number Int | Plus Expr Expr | Negate Expr
  ```

  (in the remainder of the lecture “`deriving Show`” is omitted)
- task: define a function for doubling, i.e., multiplication by 2
- solution:

  ```
  doubleNum  x = x + x      -- doubling a number
  doubleExpr e = Plus e e   -- doubling an expression
  ```
- evaluation:


doubleExpr
Plus
Negate
Number
Number
3
5
=
Plus
Plus
Plus
Negate
Number
Negate
Number
Number
3
Number
3
5
5

# Creating New Values – `Person` Example

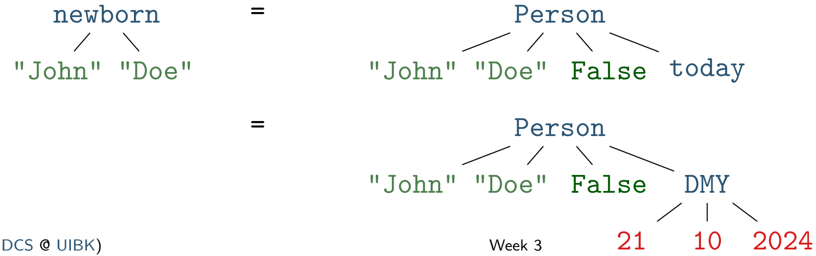
- consider `Person` datatype of last lecture

```
data Date = DMY Int Int Integer
data Person = Person String String Bool Date
```

- task: define a function that takes first- and lastname and creates a (value of type) `Person` representing a newborn with that name
- solution:

```
today = DMY 21 10 2024
newborn fName lName = Person fName lName False today
```

- evaluation



# Function Definitions using Patterns

- so far all functions definitions have been of the shape
  `funName x1 ... xN = expr`
  where `x1 ... xN` is a list of variables
- in these definitions we cannot inspect the structure of the input
- aim: define functions depending on structure of input
- example using vehicle datatype

```
data Brand = Audi | BMW | Fiat | Lamborghini
data Vehicle = Car Brand Double | Bicycle | Truck Int
```

  - task: convert a vehicle into a string
  - algorithm:
    - if the input is a car with $x$ PS, then return "a car with $x$ PS"
    - if the input is a bicycle, then return "a bicycle"
    - if the input is a truck with $x$ wheels, then return "a(n) $x$-wheel truck"
- in Haskell, structure of trees are described by patterns
- the question whether some input tree fits a pattern is called pattern matching

# Patterns

- a pattern is an expression of one of the following forms
  - `x` variable name as in a function definition
  - `_` underscore
  - `CName pat1 ... patN` constructor application with patterns `pat1 ... patN` as arguments
  - `x@pat` variable name followed by `@` and pattern

  where
  - all variables occur at most once
  - numbers, strings, and characters can be interpreted as constructors
  - parentheses might be required for nested patterns
- examples
  - `Car brand ps` ✓
  - `Car _ ps` ✓
  - `Car BMW 100` ✓
  - `Car _ (50 + 50)` + is not a constructor ✗
  - `Person "John" lName _ _` ✓
  - `p@(Person _ _ _ (DMY 21 10 _))` ✓
  - `Person name name _ _` duplicate variable ✗

# Pattern Matching

- given an expression and a pattern, pattern matching
  - determines whether the expression is covered by the shape of the pattern,
  - and in the positive case determines a substitution of pattern-variables to expressions
- examples
  - `Car brand ps` matches `expr`,
    if `expr` is an arbitrary car;
    substitution contains both brand (in variable `brand`) and horsepower (in variable `ps`)
  - `Car _ ps` matches `expr`
    if `expr` is an arbitrary car;
    substitution will substitute `ps` by the horsepower, no interest in brand
  - `Car BMW 100` matches `expr`
    if `expr` is a BMW with exactly 100 PS; substitution is empty
  - `Person "John" lName _ _` matches `expr`
    if `expr` is a person whose first name is John; substitution contains last name in `lName`
  - `p@(Person _ _ _ (DMY 21 10 _))` matches `expr`
    if `expr` is a person that can celebrate his/her birthdate today;
    substitution will contain the full person in variable `p`

## Pattern Matching Algorithm

- this slide contains an algorithm for pattern matching
- in the algorithm the substitution of variables to expressions is written as
  `x1/expr1, ..., xN/exprN`
  (here, / is not the division operator but the substitute operator)
- pattern matching algorithm for pattern `pat` and expression `expr`
  - `pat` is variable `x`: matching succeeds, substitution is `x/expr`
  - `pat` is `_`: matching succeeds, empty substitution
  - `pat` is `x@pat1`: matching succeeds if `pat1` matches `expr`;
    add `x/expr` to resulting substitution
  - `pat` is `CName pat1 ... patN`:
    - if `expr` is `OtherCName ...` with `CName` $\neq$ `OtherCName` then match fails
    - if `expr` is `CName expr1 ... exprN` then
      match `expr1` with `pat1`, ..., match `exprN` with `patN`;
      if all of these matches succeed then succeed with merged substitution, otherwise match fails
    - otherwise, first evaluate `expr` until outermost constructor is fixed
- remark: algorithm itself is described via pattern matching

## Pattern Matching Algorithm – Examples

- try to match some patterns against expression `Car BMW (20 + 80)`
  - pattern `x`: success with substitution `x / Car BMW (20 + 80)`
  - pattern `Car brand ps`: success with substitution `brand / BMW`, `ps / 20 + 80`
  - pattern `Car brand _`: success with substitution `brand / BMW`
  - pattern `Car Audi _`: failure
  - pattern `Car _ 100`: success with empty substitution, triggers evaluation
- next consider expression `Person "Liz" "Ball" True (DMY 21 10 1970)`
  - pattern `Person "John" lName _ _`: fails
  - pattern `p@(Person _ _ _ (DMY 21 10 _))`: success with substitution `p / Person "Liz" "Ball" True (DMY 21 10 1970)`

# Function Definitions with Pattern Matching

- so far all functions definitions have been of shape
  `funName x1 ... xN = expr`
- now add two generalizations
  - a function definition has the shape

    `funName pat1 ... patN = expr` $(\star)$

    where all variables in patterns `pat1 ... patN` occur at most once
  - there can be several equations for the same function
- evaluation of `funName expr1 ... exprN` via single function equation $(\star)$
  - if `pat1` matches `expr1`, `...`, `patN` matches `exprN` via some substitutions, then the equation is applicable and `funName expr1 ... exprN` is replaced by rhs `expr` with the merged substitution applied
  - otherwise, $(\star)$ is not applicable
- evaluation of `funName expr1 ... exprN`
  - apply first equation that is applicable (tried from top to bottom)
  - if no equation is applicable, abort computation with error

# Function Definitions – Example on `Person`

```
data Date = DMY Int Int Integer
data Person = Person String String Bool Date
data Option  = Some Integer | None
```

- task: change the last name of a person
  ```
  withLastName lName (Person fName _ m b) = Person fName lName m b
  ```
  remark: data is never changed but newly created

- task: compute the age of a person in years, if it is his or her birthday, otherwise return nothing
  ```
  ageYear (Person _ _ _ (DMY 21 10 y)) = Some (2024 - y)
  ageYear _   = None
  ```
  remark: here the order of equations is important

- task: create a greeting for a person
  ```
  greeting p@(Person name _ _ _) = gHelper name (ageYear p)
  gHelper n None = "Hello " ++ n
  gHelper n (Some a) = "Hi " ++ n ++ ", you turned " ++ show a
  ```
  remark: (++) concatenates two strings, `show` converts values to strings

# Merging Substitutions and Equality

- consider the following code for testing equality of two values
  ```
  equal x x = True
  equal _ _ = False
  ```
- consider evaluation of `equal 5 7`
  - first argument: `x` matches `5`, obtain substitution `x / 5`
  - second argument: `x` matches `7`, obtain substitution `x / 7`
  - merging these substitutions is not possible: `x / ???`
- Haskell avoids problem of non-mergeable substitutions by the distinct-variables-restriction in lhss, i.e., above definition is not allowed in Haskell
- correct solution for testing on equality
  - use `(==)`, a built-in operator to compare two values of the same type, the result will be of type `Bool`
  - for comparison of user-defined datatypes, replace **`deriving`** `Show` by **`deriving`** `(Show, Eq)`
  - examples: `5 == 7`, `"Peter" == name`, . . . , but not `"five" == 5`

# Function Definitions – Example on `Bool`

- consider built-in datatype `data Bool = True | False`
- consider function for conjunction of two Booleans

```
conj True  b = b
conj False _ = False
```

- example evaluation (numbers are just used as index)

```
  conj1 (conj2 True False) (conj3 True True)
-- check which equation is applicable for conj1
-- first equation triggers evaluation of first argument of conj1 (True)
-- check which equation is applicable for conj2
-- first equation is applicable with substitution b/False
= conj1 False (conj3 True True)
-- now see that only second equation is applicable for conj1
= False
```

- remark: many Boolean functions are predefined, e.g., `(&&)` (conjunction), `(||)` (disjunction), `(/=)` (exclusive-or), `not` (negation)

# Function Definitions by Case Analysis

- design principle for functions:
  define equations to cover all possible shapes of input
- example

```
data Weekday = Mon | Tue | Wed | Thu | Fri | Sat | Sun

weekend Sat = True
weekend Sun = True
weekend _   = False
```

- example: first element of a list

```
data List = Empty | Cons Integer List

first (Cons x xs) = x
first Empty       = error "first on empty list"
```

- `error` takes a string to deliver sensible error message upon evaluation
- without second defining equation, `first Empty` results in generic "non-exhaustive patterns" exception

## Recursive Function Definitions

- example: length of a list

```
len Empty = 0
len (Cons x xs) = 1 + ???  -- the length of the list xs
```

- potential problem: we would like to apply a function that we are currently defining
- this is allowed in programming and called recursion:
  a function definition that invokes itself

```
len Empty = 0
len (Cons x xs) = 1 + len xs -- len xs is recursive call
```

- make sure to have smaller arguments in recursive calls
- evaluation is as before

```
  len (Cons 1 (Cons 7 (Cons 9 Empty)))
= 1 + (len (Cons 7 (Cons 9 Empty)))
= 1 + (1 + (len (Cons 9 Empty)))
= 1 + (1 + (1 + (len Empty)))
= 1 + (1 + (1 + 0)) = 1 + (1 + 1) = 1 + 2 = 3
```

## Recursive Function Definitions – Example Append

- task: append two lists, e.g., appending $[1, 5]$ and $[3]$ yields $[1, 5, 3]$
- solution: pattern matching and recursion on first argument

```
append Empty ys = ys
append (Cons x xs) ys = Cons x (append xs ys)
```

- example evaluation

```
  append (Cons 1 (Cons 3 Empty)) (Cons 2 (Cons 7 Empty))
= Cons 1 (append (Cons 3 Empty) (Cons 2 (Cons 7 Empty)))
= Cons 1 (Cons 3 (append Empty (Cons 2 (Cons 7 Empty))))
= Cons 1 (Cons 3 (Cons 2 (Cons 7 Empty)))
```

# Recursive Function Definitions – Evaluating `Expr`

- consider datatype for expressions

```
data Expr =
    Number Integer
  | Plus Expr Expr
  | Negate Expr
```

- task: evaluate expression
- solution:

```
eval (Number x)    = x
eval (Plus e1 e2) = eval e1 + eval e2
eval (Negate e)    = - eval e
```

## Recursive Function Definitions – `Expr` to `List`

- consider datatype for expressions

```
data Expr =
    Number Integer
  | Plus Expr Expr
  | Negate Expr
```

- task: create list of all numbers that occur in expression
- solution:

```
numbers (Number x)    = Cons x Empty
numbers (Plus e1 e2) = append (numbers e1) (numbers e2)
numbers (Negate e)    = numbers e
```

## Summary

- function definitions by case analysis via pattern matching
  - patterns describe shapes of trees
  - multiple defining equations allowed, tried from top to bottom
- function definitions can be recursive
  - `funName ... = ... (funName ...) ... (funName ...) ...`
  - arguments in recursive call should be smaller than in lhs