WS 2024/2025

Functional Programming

Week 5 – Expressions, Recursion on Numbers

René Thiemann Diana Gründlinger Alexander Montag Adam Pescoller

Department of Computer Science

Last Lecture

- type variables: `a`, `b`, ... represent any type
- parametric polymorphism
  - one implementation that can be used for various types
  - polymorphic datatypes, e.g., `data List a = Empty | Cons a (List a)`
  - polymorphic functions, e.g., `append :: List a -> List a -> List a`
  - type constraints, e.g., `sumList :: Num a => List a -> a`
- predefined types: `[a]`, `Maybe a`, `Either a b`, `(a1,...,aN)`
- predefined type classes
  - arithmetic except division: `Num a`
  - arithmetic including division: `Fractional a`
  - equality between elements: `Eq a`
  - smaller than and greater than: `Ord a`
  - conversion to `String`s: `Show a`

This Lecture

- type synonyms
- expressions revisited
- recursion involving numbers

Type Synonyms

Type Synonyms

- Haskell offers a mechanism to create synonyms of types via the keyword `type`
  `type TConstr a1 ... aN = ty`
  - `TConstr` is a fresh name for a type constructor
  - `a1 ... aN` is a list of type variables
  - `ty` is a type that may contain any of the type variables
  - there is no new (value-)constructor
  - `ty` may not include `TConstr` itself, i.e., no recursion allowed

Type Synonyms – Applications, Strings

- example applications of type synonyms
  - avoid creation of new datatypes: `type Person = (String,Integer)`
  - increase readability of code

```
type Month = Int
type Day   = Int
type Year  = Int
type Date = (Day, Month, Year)

createDate :: Day -> Month -> Year -> Date
createDate d m y = (d, m, y)

-- createDate is logically equivalent to the following function,
-- but the type synonyms help to make the code more readable

createDate :: Int -> Int -> Int -> (Int, Int, Int)
createDate x y z = (x, y, z)
```

- in Haskell: `type String = [Char]`
  - in particular `"hello"` is identical to `['h', 'e', 'l', 'l', 'o']`
  - all functions on lists can be applied to `String`s as well, e.g. `(++) :: [a] -> [a] -> [a]`

Type Synonyms versus Datatypes

- type synonyms can always be encoded as separate datatype
- example encoding of persons as name and year of birth

```
type PersonTS = (String, Integer)          -- pair of name and year
data PersonDT = Person (String, Integer)   -- just add constructor Person
```

- remark: `PersonTS` and `PersonDT` are different types
  - the types `PersonTS` and `(String, Integer)` are identical
  - the type `PersonDT` is different from both `(String, Integer)` and `PersonTS`
  - `("Bob", 2002)` is of type `PersonTS`, but not of type `PersonDT`
  - `Person ("Bob", 2002)` is of type `PersonDT`, but not of type `PersonTS`
- advantages of modeling via type synonyms
  - no overhead in writing additional constructor, i.e., here `Person`
  - functions on existing types can directly be used, e.g., `fst` to access name vs.
    `name (Person p) = fst p   -- implementation for PersonDT`
- advantages of modeling via datatypes
  - separate type class instances are possible, e.g., for `show`-function (week 6)
  - possibility to hide internal representation (week 9)

Expressions Revisited

Function Definitions Revisited

- current form of function definitions

```
f :: ty                         -- optional type definition
f pat11 ... pat1M = expr1       -- first defining equation
...
f pat1M ... patNM = exprN       -- last  defining equation
```

  where expressions consist of literals, variables, and function- or constructor applications
- observations
  - case analysis only possible via patterns in left-hand sides of equations
  - case analysis on right-hand sides often desirable
  - work-around via auxiliary functions possible
  - better solution: extension of expressions

if-then-else

- most primitive form of case analysis: if-then-else
- functionality: return one of two possible results, depending on a Boolean value

```
ite :: Bool -> a -> a -> a
ite True  x y = x
ite False x y = y
```

- example application: lookup a value in a key/value-list

```
lookup :: Eq a => a -> [(a, b)] -> Maybe b
lookup x ((k, v) : ys) = ite (x == k) (Just v) (lookup x ys)
lookup _ _ = Nothing
```

- if-then-else is predefined: `if ... then ... else ...`

```
lookup x ((k, v) : ys) = if x == k then Just v else lookup x ys
```

- there is no if-then (without the else) in Haskell:
  what should be the result if the Boolean is false?
- remark: also `lookup` is predefined in Haskell;
  Prelude content (functions, (type-)constructors, type classes, ...) is typeset in `green`

Case Analysis via Pattern Matching

- observation: often case analysis is required on computed values
- implementation possible via auxiliary functions
- example: evaluation of expressions with meaningful error messages

```
data Expr a = Var String | ... -- Numbers, Addition, ...
eval :: Num a => [(String, a)] -> Expr a -> a
eval ass ...     = ...                 -- all the other cases
eval ass (Var x) = aux (lookup x ass) x -- case analysis on lookup x ass
aux (Just i) _ = i
aux _ x = error ("assignment does not include variable " ++ x)
```

- disadvantages
  - local values need to be passed as arguments to auxiliary function (here: `x`)
  - pollution of name space by auxiliary functions
    (`aux`, `aux1`, `aux2`, `auX`, `helper`, `fHelper`, `...`)
- note: if-then-else is not sufficient for above example

Case Expressions

- case expressions support arbitrary pattern matching directly in right-hand sides

```
case expr of
  pat1 -> expr1
  ...
  patN -> exprN
```

  - match `expr` against `pat1` to `patN` top to bottom
  - if `patI` is first match, then case-expression is evaluated to `exprI`
- example from previous slide without auxiliary function

```
eval ass (Var x) = case lookup x ass of
  Just i -> i
  _ -> error ("assignment does not include variable " ++ x)
```

The Layout Rule

- problem: define groups (of patterns, of function definitions, ...)
- script content is group, start nested group by **where**, **let**, **do**, or **of**
- items that start in same column are grouped together
- by increasing indentation, single item may span multiple lines
- groups end when indentation decreases
- **ignore layout:** enclose groups in '{' and '}' and separate items by ';'

Examples

with layout:

```
and b1 b2 = case b1 of
  True -> case b2 of
    True -> True
    False -> False
  False -> False
```

without layout:

```
and b1 b2 = case b1 of
  { True -> case b2 of
  { True -> True; False -> False };
    False -> False }
```


White-Space in Haskell

- because of layout rule, white-space in Haskell matters (in contrast to many other programming languages)
- avoid tabulators in Haskell scripts (tab-width of editor versus Haskell-compiler)

Example

```
and1 b1 b2 = case b1 of
  True -> case b2 of
    True -> True
    False -> False
```

```
and2 b1 b2 = case b1 of
  True -> case b2 of
    True -> True
  False -> False
```

```
ghci> and1 True False
False

ghci> and2 True False
*** error: non-exhaustive patterns
```


The let Construct

- **let**-expressions are used for local definitions
- syntax

```
let
  pat                 = expr  -- definition by pattern matching
  fname pat1 ... patN = expr  -- function definition
in expr                       -- result
```

- each **let**-expression may contain several definitions (order irrelevant)
- definitions result in new variable-bindings and functions
  - may be used in every expression expr above
  - are not visible outside **let**-expression

Number of Real Roots via let Construct

```
-- Prelude type and function for comparing two numbers
data Ordering = EQ | LT | GT
compare :: Ord a => a -> a -> Ordering

-- task: determine number of real roots of ax^2 + bx + c
numRoots a b c = let
    disc = b^2 - 4 * a * c   -- local variable
    analyse EQ = 1           -- local function
    analyse LT = 0
    analyse GT = 2
  in analyse (compare disc 0)
```

The `where` Construct

- `where` is similar to `let`, used for local definitions
- syntax

```
f pat1 .. patM = expr                -- defining equation (or case)
  where  pat                 = expr  -- pattern matching
         fname pat1 .. patN = expr   -- function definitions
```

- each `where` may consist of several definitions (order irrelevant)
- local definitions introduce new variables and functions
  - may be used in every expression `expr` above
  - are not visible outside defining equation / case-expression
- remark: in contrast to `let`, when using `where` the defining equation of `f` is given first

```
numRoots a b c = analyse (compare disc 0) where
    disc = b^2 - 4 * a * c    -- local variable
    analyse EQ = 1            -- local function
    analyse LT = 0
    analyse GT = 2
```


Guarded Equations

- defining equations within a function definition can be guarded
- syntax:

```
fname pat1 ... patM
  | cond1 = expr1
  | cond2 = expr2
  | ...
  where ... -- optional where-block
```

  where each `condI` is a Boolean expression
- whenever `condI` is first condition that evaluates to `True`, then result is `exprI`
- next defining equation of `fname` considered, if no condition is satisfied

```
numRoots a b c
  | disc > 0   = 2
  | disc == 0  = 1
  | otherwise  = 0              -- otherwise = True
  where disc = b^2 - 4 * a * c -- disc is shared among cases
```


Example: Roots

- task: compute the sum of the roots of a quadratic polynomial
- solution with potential runtime errors

```
roots :: Double -> Double -> Double -> (Double, Double)
roots a b c
  | a == 0 = error "not quadratic"
  | d <  0 = error "no real roots"
  | otherwise = ((- b - r) / e, (- b + r) / e)
  where d = b * b - 4 * a * c
        e = 2 * a
        r = sqrt d

sumRoots :: Double -> Double -> Double -> Double
sumRoots a b c = let
    (x, y) = roots a b c  -- pattern match in let
  in x + y
```

- note: non-variable patterns in `let` are usually only used if they cannot fail; otherwise, use `case` instead of `let`

Example: Roots (Continued)

- task: compute the sum of the roots of a quadratic polynomial
- solution with explicit failure via `Maybe`-type

```
roots :: Double -> Double -> Double -> Maybe (Double, Double)
roots a b c
  | a == 0 = Nothing
  | d <  0 = Nothing
  | otherwise = Just ((- b - r) / e, (- b + r) / e)
  where d = b * b - 4 * a * c
        e = 2 * a
        r = sqrt d

sumRoots :: Double -> Double -> Double -> Maybe Double
sumRoots a b c =
  case roots a b c of          -- case for explicit error handling
    Just (x, y) -> Just (x + y) -- nested pattern matching
    n -> Nothing                -- can't be replaced by n -> n! (types)
```

Recursion on Numbers

Recursion on Numbers

- recursive function
  ```
  f pat1 ... patN = ... (f expr1 ... exprN) ...
  ```
  where input arguments should somehow be larger than arguments in recursive call:
  ```
  (pat1, ..., patN) > (expr1, ..., exprN)   -- for some relation >
  ```
- decrease often happens in one specific argument (the i-th argument always gets smaller)
- so far the decrease in size was always w.r.t. tree size
  - length of list gets smaller
  - arithmetic expressions (`Expr`) are decomposed, i.e., number of constructors is decreased
- if argument is a number (tree size is always 1), then still recursion is possible; example: the value of number might decrease
- frequent cases
  - some number i is decremented until it becomes 0 (while $i \neq 0 \ldots i := i - 1$)
  - some number i is incremented until it reaches some bound n (while $i < n \ldots i := i + 1$)

Example: Factorial Function

- mathematical definition: $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$, $0! = 1$
- implementation D: count downwards
  ```
  factorial :: Integer -> Integer
  factorial 0 = 1
  factorial n = n * factorial (n - 1)
  ```
  - in every recursive call the value of `n` is decreased
  - `factorial n` does not terminate if `n` is negative (hit Ctrl-C in ghci to stop computation)
- implementation U: count upwards, use accumulator (here: `r` stores accumulated (r)esult)
  ```
  factorial :: Integer -> Integer
  factorial n = fact 1 1 where
    fact r i
      | i <= n = fact (i * r) (i + 1)
      | otherwise = r
  ```
  - in every recursive call the value of `n - i` is decreased
  - implementation U is equivalent to imperative program (with local variables `r` and `i`)

Example: Combined Recursion

- recursion on trees and numbers can be combined
- example: compute the n-th element of a list
  ```
  nth :: [a] -> Int -> a
  nth (x : _)  0 = x                -- indexing starts from 0
  nth (_ : xs) n = nth xs (n - 1)   -- decrease of number and list-length
  nth _ _ = error "no nth-element"
  ```
- example: take the first n-elements of a list
  ```
  take :: Int -> [a] -> [a]
  take _ [] = []
  take n (x : xs)
    | n <= 0 = []
    | otherwise = x : take (n - 1) xs -- decrease of number and list-length
  ```
- remarks
  - both `take` and n-th element (`!!`) are predefined
  - `drop` is predefined function that removes the first n-elements of a list
  - equality: `take n xs ++ drop n xs == xs`

Example: Creating Ranges of Values

- task: given lower bound l and upper bound u, compute list of numbers $[l, l+1, \ldots, u]$
- algorithm: increment l until $l > u$ and always add l to front of list
```
range l u
  | l <= u = l : range (l + 1) u
  | otherwise = []
```
- remark: (a generalized version of) `range l u` is predefined and written `[l .. u]`
- example: concise definition of factorial function
  - `factorial n = product [1 .. n]`
    where `product :: Num a => [a] -> a` computes the product of a list of numbers

Summary

- type synonyms via **`type`**
- expressions with local definitions and case analysis
- recursion on numbers