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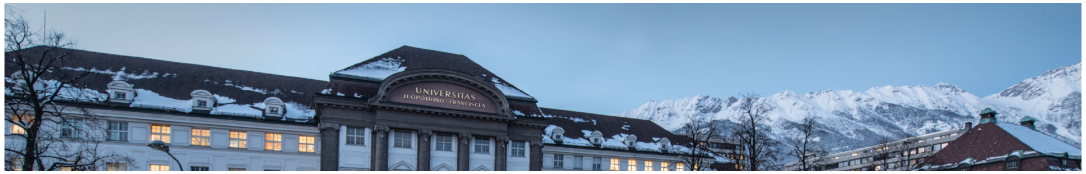


# Advanced Functional Programming

## Week 1 – Organisation and Introduction, Strict- and Lazy-Evaluation

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# Organization



## Organization of Course

- LV-Number: 703139
- lecturer: René Thiemann
  consultation hours: Tuesday 10:15 – 11:15 in 3M09 (ICT building)
- time and place: Wednesday, 11:00 – 13:30 in SR 12
- course website: `http://cl-informatik.uibk.ac.at/teaching/ws25/afp/`
- lecture will be in English
- slides are available online and contain links
- modus: VU 3
  - 3 hours per week
  - attendance is obligatory
  - VU: lecture and exercises combined
    - today: just lecture
    - from next week onwards: first, presentation of exercises; afterwards lecture



## Schedule

- detailed schedule: see website
- special dates
  - today: just lecture
  - January 21, Q & A session, no new content
  - January 28: exam

## Evaluation

- 50 % exercises + 50 % exam
- exam on January 28
- exercises will be handed out every week
- mark solved exercises and upload Haskell sources in OLAT
- deadline in OLAT: Tuesday, 4pm
- definition of solved:
  - 100 % solutions are not required, but a significant part of tasks should have been solved
  - capability to explain your solution to everyone in this room
  - not permitted: just copy some internet/chatGPT solution without understanding it
- positive evaluation: get in total at least 50 % of points



## Literature

- slides and exercises
  - no other topics will appear in exam ...
  - ... but topics need to be understood thoroughly
    - read and write functional programs
    - apply presented techniques on new examples
    - not only knowledge reproduction
- Bryan O'Sullivan, John Goerzen and Don Stewart. Real World Haskell, O'Reilly.
- ... see slides



## Prerequisites: Basic Knowledge of Functional Programming

- knowledge on lists, trees and other algebraic data types
- knowledge on recursive function definitions
- basic knowledge on type-classes (`Eq`, `Ord`, `Show`, `Num`)
- basic knowledge on programming with higher-order functions
  (`map`, `filter`, `foldr`, `.`, partial application, ...)
- basic knowledge on `IO`
  (separate pure from IO-computations, do-notation, ...)



# Strict- and Lazy-Evaluation

## Example

Consider

- program `square x = x * x`, and
- expression `square (3 + 2)`

## Different Ways to Apply Equations

- strict/innermost: evaluate arguments before doing a function application

  `square (3 + 2) = square 5 = 5 * 5 = 25`

- non-strict/lazy: apply program equation as soon as possible

  `square (3 + 2) = (3 + 2) * (3 + 2) = 5 * 5 = 25`

  where the sub-expression 3+2 is shared and hence, only evaluated once



## Values and Thunks

- value: a fully evaluated term, e.g., `5`, `"hello"`, `[1,2,3]`
- thunk: a term that needs further evaluation, e.g., `2 + 3`, `"hel" ++ "lo"`, ...
- strict/innermost: evaluate arguments to values before invoking function application
- non-strict/lazy: arguments can be passed as values or as thunks
- consequences
  - strict/innermost is easier to implement; takes less space per cell
  - non-strict/lazy includes overhead when working with thunks; admits new kinds of programming styles
- ML and OCaml use a strict/innermost evaluation strategy
- Haskell uses non-strict/lazy as default evaluation strategy; strict/innermost on demand
  - offer strict and lazy folding functions
  - offer strict and lazy arrays
  - offer strict and lazy dictionaries
  - enforce strictness via `seq`, via strict datatypes, ...



## Example (`foldl` and `foldl'`)

```
foldl, foldl' :: (b -> a -> b) -> b -> [a] -> b
foldl f y [] = y
foldl f y (x : xs) =
  let z = f y x
  in foldl f z xs

foldl' f y [] = y
foldl' f y (x : xs) =
  let z = f y x
  in seq z $ foldl' f z xs
```

## Remark

- `seq x y` returns `y` after evaluating `x` to weak-head normal form (WHNF), i.e., after outermost constructor has been computed
- example:

  `(let xs = take 2 [5..] in seq xs xs) = ... = 5 : take (2 - 1) [5 + 1..]`



## Example (Lazy Evaluation via `foldl`)

```
foldl f y [] = y
foldl f y (x : xs) =
  let z = f y x
  in foldl f z xs

  foldl (+) 0 [1,2,3,4,5,6]
= let z1 = 0 + 1 in foldl (+) z1 [2,3,4,5,6]
= let z1 = 0 + 1 in let z2 = z1 + 2 in foldl (+) z2 [3,4,5,6] = ...
= let z1 = 0 + 1 in let z2 = z1 + 2 in let z3 = z2 + 3 in
  let z4 = z3 + 4 in let z5 = z4 + 5 in let z6 = z5 + 6 in foldl (+) z6 []
= let z1 = 0 + 1 in let z2 = z1 + 2 in let z3 = z2 + 3 in
  let z4 = z3 + 4 in let z5 = z4 + 5 in let z6 = z5 + 6 in z6
= ((((0 + 1) + 2) + 3) + 4) + 5) + 6
= ... = 21
```

Linear space requirement!

### Example (Strict Evaluation via `foldl'`)

```
foldl' f y [] = y
foldl' f y (x : xs) =
  let z = f y x
  in seq z $ foldl' f z xs

  foldl' (+) 0 [1,2,3,4,5,6]
= let z1 = 0 + 1 in seq z1 $ foldl' (+) z1 [2,3,4,5,6]
= let z1 = 1 in seq z1 $ foldl' (+) z1 [2,3,4,5,6]
= foldl' (+) 1 [2,3,4,5,6]
= let z2 = 1 + 2 in seq z2 $ foldl' (+) z2 [3,4,5,6]
= let z2 = 3 in seq z2 $ foldl' (+) z2 [3,4,5,6]
= foldl' (+) 3 [3,4,5,6]
= ...
= foldl' (+) 21 []
= 21
```

Constant space requirement!



### Example (Sometimes `foldl` is Preferable)

```
mulNS x 0 = 0
mulNS x y = x * y

-- compare
foldl mulNS 1 [3,6,undefined,0,7]
-- with
foldl' mulNS 1 [3,6,undefined,0,7]

-- result: only the former succeeds
```



## Use `seq` Carefully

- `seq` forces only an evaluation, if `seq` itself is at a position which should be evaluated
- usually, put `seq` on the outside

```
f 0 y = ...
f x y = let z = ... in z `seq` f (x - 1) z   -- evaluation of z to WHNF
f x y = let z = ... in f (x - 1) (z `seq` z) -- no effect
f x y =
  let x1 = x - 1;
      z = ...
  in x1 `seq` z `seq` f x1 z      -- evaluate both x1 and z to WHNF
                                  -- here: useless for x1
```



## Benefits from Lazy Evaluation: Modularity

- composing several programs can work out nicely with lazy evaluation, where strict evaluation is not performant
- example: compute the ten smallest elements in a list `xs`
- lazy approach: `take 10 (sort xs)`
  - approach can be efficient, since due to laziness, not all of `sort xs` has to be computed (efficiency depends on utilized sorting algorithm)
- strict approach
  - `take 10 (sort xs)` is inefficient to evaluate, if `xs` is long
  - writing separate program from scratch requires work

## Programming with Lazy Evaluation

- task
  - replace all elements in a non-empty list by the minimum in the list ...
  - ... with only one list-traversal
- solution

```
findMinRepl :: Ord b => a -> [b] -> (b, [a])
findMinRepl r [x] = (x, [r])
findMinRepl r (x : xs) = case findMinRepl r xs of
  (m, ys) -> (min m x, r : ys)

replAllByMin :: Ord a => [a] -> [a]
replAllByMin xs =
  let (m, ys) = findMinRepl m xs
  in ys
```

- trick: `m` is evaluated lazily in `replAllByMin`



## Programming with Lazy Evaluation

```
findMinRepl r [x] = (x, [r])
findMinRepl r (x : xs) = case findMinRepl r xs of
  (m, ys) -> (min m x, r : ys)
replAllByMin xs = let (m, ys) = findMinRepl m xs in ys

  replABM [2,6,1]
= let (m, ys) = fMR m [2,6,1] in ys
= let (m, ys) = case fMR m [6,1] of (m1, ys1) -> (min m1 2, m : ys1) in ys
= let (m, ys) = case (case fMR m [1] of (m2, ys2) -> (min m2 6, m : ys2))
     of (m1, ys1) -> (min m1 2, m : ys1) in ys
= let (m, ys) = case (case (1, [m]) of (m2, ys2) -> (min m2 6, m : ys2))
     of (m1, ys1) -> (min m1 2, m : ys1) in ys
= let (m, ys) = case (min 1 6, [m, m])
     of (m1, ys1) -> (min m1 2, m : ys1) in ys
= let (m, ys) = (min (min 1 6) 2, [m, m, m]) in ys
= [min (min 1 6) 2, min (min 1 6) 2, min (min 1 6) 2] = ... = [1, 1, 1]
```



## Programming with Lazy Evaluation – Lazy Arrays

- several container data structures (arrays, dictionaries, ...) are provided both in a strict and in a lazy variant in Haskell libraries
- advantage of strict versions
  - no overhead from working with thunks
  - less memory consumption, no boxing and unboxing of values
- advantage of lazy versions
  - lazy initialization becomes possible:
    already consume parts of array during construction (similar to `m` in previous example)
- documentation
  - https://hackage.haskell.org/package/array/docs/Data-Array-IArray.html
  - https://hackage.haskell.org/package/array/docs/Data-Array-Unboxed.html



## Example with Lazy Initialization

```
import qualified Data.Array.IArray as L  -- lazy, boxed, immutable arrays

fibsLazyArray :: Int -> [Integer]
fibsLazyArray n =
  let a :: L.Array Int Integer
      a = L.genArray (0,n)
         (\ i -> if i <= 1 then 1 else a L.! (i - 1) + a L.! (i - 2))
  in L.elems a

-- lazy approach: in order to construct array a, we already use it

-- index types Ix might be Int, Integer, Char, (Int, Int), ...
-- L.genArray :: (IArray a e, Ix i) => (i, i) -> (i -> e) -> a i e
-- (L.!) :: (IArray a e, Ix i) => a i e -> i -> e
-- L.elems :: (IArray a e, Ix i) => a i e -> [e]
```

## Lazy Initialization does Not Work with Strict Arrays

```haskell
import Data.Array.Unboxed as S -- strict, unboxed arrays
-- UArray can store elements of type Int, Word32, ...,
--   but not Integer, String, ...
import Data.Word (Word64)

fibsStrictArray :: Int -> [Word64]
fibsStrictArray n =
  let a :: S.UArray Int Word64
      a = S.genArray (0,n)
          (\ i -> if i <= 1 then 1 else a S.! (i - 1) + a S.! (i - 2))
  in S.elems a

-- computation of fibsStrictArray 10 does not succeed

-- similar interface in comparison to lazy arrays
-- S.genArray :: (S.IArray a e, S.Ix i) => (i, i) -> (i -> e) -> a i e
```



## Another Example for Lazy Containers: Dynamic Programming

- bracketing problem
  - given is list of $n-1$ compatible matrices $A_0\ A_1\ \dots A_{n-2}$
  - in fact, only the dimensions of $A_i$ are given: $[a_0, \dots, a_{n-1}]$, $A_i$ has dimension $a_i \times a_{i+1}$
  - task: figure out cheapest way to multiply all matrices, e.g., $(A_0 A_1)(A_2(A_3 A_4))$
  - algorithm computes optimal costs to multiply $A_i \dots A_j$
  - $cost(i,i) = 0$
  - $cost(i,j) = \min\{cost(i,k) + cost(k+1,j) + \underbrace{a_i a_{k+1} a_{j+1}}_{\text{matrix-multiplication}} \mid i \le k < j\}$ if $i < j$

$$A_i \dots A_j = \underbrace{(A_i \dots A_k)}_{a_i \times a_{k+1}} \underbrace{(A_{k+1} \dots A_j)}_{a_{k+1} \times a_{j+1}}$$

- naive recursive computation of $cost$ results in exponential algorithm
- solution: dynamic programming
  - compute values of $cost(i,j)$ for increasing differences of $i$ and $j$ – without recomputation



## Lazy Maps and Sets

- `Data.Map.Lazy` provides lazy dictionaries (or: maps) in Haskell
- multiple construction possibilities
  - `empty :: Map k v`
  - `insert :: Ord k => k -> v -> Map k v -> Map k v`
  - `unionWith :: Ord k => (v -> v -> v) -> Map k v -> Map k v -> Map k v`
  - `fromList :: Ord k => [(k, v)] -> Map k v`
- querying single keys
  - `lookup :: Ord k => k -> Map k v -> Maybe v` (optional value)
  - `! :: Ord k => Map k v -> k -> v` (might throw error)
- implemented as balanced trees
- `Data.Set` has similar functionality to represent sets
- documentation
  - https://hackage.haskell.org/package/containers/docs/Data-Map-Lazy.html
  - https://hackage.haskell.org/package/containers/docs/Data-Map-Strict.html
  - https://hackage.haskell.org/package/containers/docs/Data-Set.html



## Implementation of Bracketing Problem in Haskell via Lazy Maps

```haskell
import qualified Data.Array.IArray as L
import qualified Data.Map.Lazy as M -- lazy dictionaries

optBracketCosts :: [Integer] -> Integer
optBracketCosts xs =
  let n = length xs - 1
      a = L.listArray (0,n) xs :: L.Array Int Integer
      m = M.fromList [((i,j),cost i j) | i <- [0..n - 1], j <- [i..n-1]]
      cost i j
        | i == j = 0
        | otherwise = minimum [costSplit k | k <- [i .. j - 1]] where
          costSplit k =
            let c1 = m M.! (i,k)
                c2 = m M.! (k+1,j)
            in c1 + c2 + a L.! i * a L.! (k + 1) * a L.! (j + 1)
  in cost 0 (n-1)
```

## Analysis of `optBracketCosts`

- no explicit sequence is given, in which dictionary is filled
- instead, an over-approximation of required values `(i,j)` is used:
  `i <- [0..n - 1], j <- [i..n-1]`
- recursion is done implicitly: from `(i,j)` with `i <= k <= j - 1`
  invoke both `(i,k)` and `(k+1,j)`
- input list `xs` is converted to array `a` for efficient element access
- the array might be changed to strict version (if input would be `[Int]`),
  but the dictionary must be lazy



## Comparison of Maps and Immutable Arrays in Haskell

- lookup is logarithmic for maps, but constant time for arrays
- keys are arbitrary ordered objects, whereas type of array indices is restricted
- keys can have arbitrary gaps, whereas indices in arrays are dense
- maps also support deletion and change of key-value pairs
- both are available in strict and lazy version
- several variants of maps are available
  https://haskell-containers.readthedocs.io/en/latest/map.html



## A Note on the Haskell Sources

- the demos and exercises are provided as a cabal package
- make sure to have ghc and cabal installed (via package manager or via ghcup);
  all programs are tested with ghc version 9.12.2
- download and extract the sources from the AFP website
- change directory into afp_01 (where `afp.cabal` is located)
- workflow for exercise sheet 1
  - `cabal repl` (run cabal project interactively)
  - `:m Exercise01` (load module Exercise01.hs)
  - `do {testsBrackets; testsEmb}` (run tests)
  - (edit `src/Exercise01.hs`)
  - `:r` (reload program after changes)
  - note: on first run, lean-check and other packages might be installed
  - just upload updated version of `Exercise01.hs` in OLAT



## Literature

- Real World Haskell, pages 32–33, 108–110, 270–274, 289–292